

An explicit model for the adiabatic evolution of quantum observables driven by 1D shape resonances.

A. Faraj*, A. Mantile*, F. Nier*

Dedicated to the memory of P. Duclos.

Abstract

This paper is concerned with a linearized version of the quantum transport problem where the Schrödinger-Poisson operator is replaced by a non-autonomous Hamiltonian, slowly varying in time. We consider an explicitly solvable system where a semiclassical island is described by a flat potential barrier, while a time dependent 'delta' interaction is used as a model for a single quantum well. Introducing, in addition to the complex deformation, a further modification formed by artificial interface conditions, we give a reduced equation for the adiabatic evolution of the sheet density of charges accumulating around the interaction point.

1 Introduction

The derivation of reduced models for the dynamics of transverse quantum transport with concentrated non-linearities plays a central role in the mathematical analysis of semiconductor heterostructures like tunneling diodes or possibly more complex structures. The conduction band edge-profile of such systems has been described using Schrödinger-Poisson Hamiltonians with quantum wells in a semiclassical island, where a non-linear potential term, depending on the local charge density, approximates in the mean field limit the repulsive interaction between the charge carriers. A functional framework for such a model is proposed in [16], based on Mourre's theory and Sigal-Soffer propagation estimates, and implements a dynamical nonlinear version of the Landauer-Büttiker approach. The analysis of the related steady state problem, developed in [6], [7], [17] on the basis of the Helffer-Sjöstrand approach to resonances [13], has provided with an asymptotic reduced equation for the nonlinear potential, which elucidates the influence of the geometry of the potential on the feasibility of hysteresis phenomena, already studied in [14], [18], and confirms the general belief arising in physical literature: The nonlinear phenomena are governed by a finite number of resonant states.

For the dynamical problem, we conjecture that the nonlinear dynamics follows the time evolution of those resonant states corresponding to shape resonances which are asymptotically embedded in some relevant energy interval when the quantum scale of the problem, parametrized by \hbar , goes to zero. It is known, at least in the linear case, that this evolution shows an exponential decay behaviour having physical interpretation in terms of truncated resonant states (lying in L^2). The *quasi-resonant states* concentrate their mass inside the quantum well's support – the classical region of motion of our model – on a long time scale given by the inverse of the imaginary part of the resonant energies E_{res}^h . In this connection, the Poisson potential, as well as the charge density for the nonlinear modelling, are expected to evolve slowly in time, with an adiabatic parameter ε which is related to the quantum scale of the system according to: $\varepsilon = \mathcal{O}(\text{Im } E_{res}^h) \sim e^{-\frac{\tau}{\hbar}}$, for some $\tau > 0$.

This paper is concerned with a linearized version of the transport problem where the Schrödinger-Poisson operator is replaced by a non-autonomous Hamiltonian, slowly varying in time, and whose time profile takes into account the evolution of the nonlinear potential. This allows us to separate

*IRMAR, UMR - CNRS 6625, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France.

the adiabatic evolution generated by the double scale Hamiltonian, from questions concerned with the nonlinear nature of the original problem. In particular, we consider an explicitly solvable model where the semiclassical island is described by a flat potential barrier, while a time dependent 'delta' interaction describes, with the suitable scaling, a single quantum well. Our approach consists in introducing, in addition to the complex deformation, a further modification formed by artificial interface conditions. According to the results obtained in [11], an adiabatic theorem holds for this modified system (see Theorem 7.1 in [11]), while small perturbations are produced on the relevant spectral quantities (actually the same remains true under more general assumptions). In this simplified framework, we give a reduced equation for the adiabatic evolution of the sheet density of charges accumulating around the interaction point. This result is coherent with the reduced model predicted in [18],[19]. Moreover, some corrections arise, depending on the time profile of the perturbation, which can be relevant in realistic physical situations.

2 The model

We consider the time evolution of a quantum observable for a family of non-selfadjoint Hamiltonians adiabatically depending on the time. Our model is defined by the Schrödinger operators $H_{\theta_0, \alpha(t)}^h$

$$H_{\theta_0, \alpha(t)}^h = -h^2 \Delta_{\theta_0} + 1_{(a,b)} V_0 + h\alpha(t) \delta_c, \quad (2.1)$$

where Δ_{θ_0} is a singularly perturbed Laplacian with artificial interface conditions on the boundary of $\mathbb{R} \setminus \{a, b\}$

$$\begin{cases} D(\Delta_{\theta_0}) = \left\{ u \in H^2(\mathbb{R} \setminus \{a, b\}) : \begin{cases} e^{-\frac{\theta_0}{2}} u(b^+) = u(b^-); & e^{-\frac{3}{2}\theta_0} u'(b^+) = u'(b^-) \\ e^{-\frac{\theta_0}{2}} u(a^-) = u(a^+); & e^{-\frac{3}{2}\theta_0} u'(a^-) = u'(a^+) \end{cases} \right\} \\ \Delta_{\theta_0} u = \partial_x^2 u. \end{cases} \quad (2.2)$$

Meanwhile $1_{(a,b)} V_0 + h\alpha \delta_c$ is a selfadjoint time dependent point interaction defined with: $V_0 > 0$, $c \in (a, b)$, $\alpha \in C^\infty(0, T)$ and requiring the condition

$$u \in H^2((a, b) \setminus \{c\}) \cap H^1(a, b), \quad h[u'(c^+) - u'(c^-)] = \alpha(t)u(c), \quad (2.3)$$

for all $u \in D(H_{\theta_0, \alpha(t)}^h)$ (we refer to [1] for the definition of delta interaction Hamiltonians).

An accurate analysis of this class of operators has been given, in [11]. It is shown that the interface conditions introduce small errors, controlled by θ_0 , with respect to the original selfadjoint model (defined by $\theta_0 = 0$). The main interest in introducing the artificial perturbation Δ_{θ_0} rests upon the fact that the corresponding Hamiltonian defines, under complex deformation, a dynamical systems of contractions. This provides us with an alternative approach to the adiabatic evolution of the shape resonances possibly associated with our model, which can be treated in terms of (adiabatic evolution of) spectral projectors for the non-selfadjoint deformed operator (for this point, we refer to Theorem 7.1 in [11]; see also the work of A. Joye [15] for the adiabatic evolution of dynamical systems without uniform time estimates on the semigroup).

Let us consider a positive smooth function χ , $\text{supp } \chi \subset (a, b)$; in our framework χ is the quantum observable associated with the charge density accumulated in a small neighbourhood of the quantum well. The expected value of this density sheet is associated with

$$A_{\theta_0}(t) = \text{Tr} [\chi \rho_t^h], \quad (2.4)$$

where ρ_t^h is the time evolution of the density operator. The initial state of the system,

$$\rho_0^h = \int \frac{dk}{2\pi h} g(k) |\psi_-(k, \cdot, \alpha_0)\rangle \langle \psi_-(k, \cdot, \alpha_0)|, \quad (2.5)$$

is defined by a superposition of incoming scattering states solving

$$(H_{\theta_0, \alpha}^h - k^2) \psi_-(k, \cdot, \alpha) = 0,$$

according to the out-of-equilibrium assumption $g = 1_{\mathbb{R}_+} g$. Using an adiabatic approximation for the time variations of the coupling parameter α , ρ_t^h writes as

$$\rho_t^h = \int \frac{dk}{2\pi h} g(k) |u(k, \cdot, t)\rangle \langle u(k, \cdot, t)|, \quad (2.6)$$

with

$$\begin{cases} i\varepsilon \partial_t u(k, \cdot, t) = H_{\theta_0, \alpha(t)}^h u(k, \cdot, t), \\ u_{t=0} = \psi_-(k, \cdot, \alpha_0). \end{cases} \quad (2.7)$$

Adiabatic dynamics have already been considered within the modelling of out-of-equilibrium quantum transport in [3], [4], [9], playing with the continuous spectrum with selfadjoint techniques. For energies close to the shape resonances, the relevant observable of this problem follow the adiabatic evolution of resonant states. Then, a different approach consists in using complex deformations, originally introduced in [2], [5]. In [11], we define a family of exterior complex deformations U_θ for Hamiltonians with compactly supported potentials in (a, b)

$$U_\theta u(x) = \begin{cases} e^{\frac{\theta}{2}} u(e^\theta(x-b) + b), & x > b, \\ u(x), & x \in (a, b), \\ e^{\frac{\theta}{2}} u(e^\theta(x-a) + a), & x < a. \end{cases} \quad (2.8)$$

The corresponding deformed operator is obtained by conjugation: $H_{\theta_0, \alpha(t)}^h(\theta) = U_\theta H_{\theta_0, \alpha(t)}^h U_\theta^{-1}$. It is explicitly written as

$$H_{\theta_0, \alpha(t)}^h(\theta) = -h^2 e^{-2\theta 1_{\mathbb{R} \setminus (a,b)}} \Delta_{\theta_0 + \theta} + 1_{(a,b)} V_0 + h\alpha(t) \delta_c. \quad (2.9)$$

Since χ commutes with U_θ for all values $\theta \in \mathbb{C}$, the variable $A_{\theta_0}(t)$ can be defined in terms of deformed quantities. Thus $A_{\theta_0}(t)$ can be rephrased as

$$A_{\theta_0}(t) = \text{Tr} [U_\theta^* \chi U_\theta \rho_t^h] = \text{Tr} [\chi U_\theta \rho_t^h U_\theta^*]. \quad (2.10)$$

Denoting with $S_{\theta_0, \varepsilon}(t, s)$ the time propagator related to $\frac{1}{\varepsilon} H_{\theta_0, \alpha(t)}^h$, we get

$$A_{\theta_0}(t) = \text{Tr} [\chi U_\theta S_{\theta_0, \varepsilon}(t, 0) \rho_0^h S_{\theta_0, \varepsilon}^*(t, 0) U_\theta^*] = \text{Tr} [\chi U_\theta S_{\theta_0, \varepsilon}(t, 0) U_\theta^{-1} U_\theta \rho_0^h U_\theta^* (U_\theta^{-1})^* S_{\theta_0, \varepsilon}^*(t, 0) U_\theta^*],$$

where $(U_\theta^{-1})^* = (U_\theta^*)^{-1}$ is used. The conjugation: $U_\theta S_{\theta_0, \varepsilon}(t, 0) U_\theta^{-1}$ defines the propagator associated with the deformed Hamiltonian $\frac{1}{\varepsilon} H_{\theta_0, \alpha(t)}^h(\theta)$. Thus (2.4) reformulates as follows

$$A_{\theta_0}(t) = \text{Tr} [\chi \rho_t^h(\theta)] , \quad (2.11)$$

$$\rho_t^h(\theta) = \int \frac{dk}{2\pi h} g(k) |u_\theta(k, \cdot, t)\rangle \langle u_\theta(k, \cdot, t)|, \quad (2.12)$$

with

$$\begin{cases} i\varepsilon \partial_t u_\theta(k, \cdot, t) = H_{\theta_0, \alpha(t)}^h(\theta) u_\theta(k, \cdot, t) \\ u_{t=0} = U_\theta \psi_-(k, \cdot, \alpha_0) \end{cases} . \quad (2.13)$$

We will consider this evolution problem under the following assumptions:

h1) The deformation and the interface conditions parameters are equals and

$$\theta = \theta_0 = ih^{N_0}, \quad N_0 > 2 \quad (2.14)$$

h2) The time dependent coupling parameter α_t is a $C^\infty(0, T)$ real valued function with compact range in $(-2V_0^{\frac{1}{2}}, 0)$ and such that:

i) Its first variations have size h , i.e.:

$$\forall s, t \in [0, T] \Rightarrow |\alpha_t - \alpha_s| \leq \frac{2h}{\sqrt{V_0} d_0} \quad (2.15)$$

where $d_0 > 0$ is specified further.

ii) There exists a positive integer J such that the vector $\left\{ \partial_t^j \alpha(t) \right\}_{j=1}^J$ is not null for all t .

h3) The initial state is defined with a smooth and compactly supported partition function g such that:

$$\text{supp } g(k) = \left\{ k > 0, \quad |k^2 - \lambda_0| < 2\frac{h}{d_0} \right\} \quad (2.16)$$

where λ_0 denotes some asymptotic energy: $\lambda_0 \in (0, V_0)$, while d_0 and h_0 are such that: $\text{supp } g \subset (0, V_0)$ uniformly w.r.t. $h \in (0, h_0)$.

Furthermore, we assume that $g(E^{\frac{1}{2}})$ extends to an holomorphic function of E in the complex neighbourhood of λ_0 of radius $\frac{h}{d_0}$.

The function $\chi \in C_0^\infty(a, b)$ is real valued and such that:

$$\begin{cases} \chi = (c - 2\eta, c + 2\eta) \\ \chi(x)|_{x \in (c-\eta, c+\eta)} = 1 \end{cases}, \quad \eta < d(c, \{a, b\}) \quad (2.17)$$

$d(c, \{a, b\})$ denoting the distance of c from the the boundary of the interval (a, b) .

h4) The adiabatic parameter is fixed to the exponential scale defined by

$$\varepsilon = e^{-\frac{|\alpha_0|}{h} d(c, \{a, b\})}, \quad (2.18)$$

The explicit character of our model and the adiabatic theorem, obtained in [11] for this class of non-selfadjoint Hamiltonians, allow to obtain a complete description of the asymptotic behaviour of $A_{\theta_0}(t)$ as $h \rightarrow 0$, in particular concerned with the position of the delta shaped potential well. To formulate our results, we adopt the following notation.

Notation a) The resonance at time t and the related resonant state are respectively denoted with $E(t)$ and $G(t)$.

b) The expression: $X_\varepsilon = \tilde{\mathcal{O}}(\varepsilon^n)$ is used for the following condition: $\forall \delta \in (0, 1)$, there exists $C_{X, \delta}$ such that

$$|X_\varepsilon| \leq C_{X, \delta} \varepsilon^{n-\delta}. \quad (2.19)$$

Theorem 2.1 Let $\lambda_t = V_0 - \frac{\alpha_t^2}{4}$, $t \in [0, T]$ and assume the conditions (h1)-(h4) to hold with: $h \in (0, h_0)$, h_0 small, $\lambda_0 = V_0 - \frac{\alpha_0^2}{4}$, and $d > 0$ such that: $\lambda_t^{\frac{1}{2}} \in \text{supp } g \subset (0, V_0)$ for all t . The following conditions hold:

i) For any $t \in [0, T]$, there exists a single resonance, $E(t)$, of $H_{\theta_0, \alpha(t)}^h$ such that: $\text{Re } E^{\frac{1}{2}}(t) \in (0, V_0)$. With the notation: $E(t) = E_R(t) - i\Gamma_t$, the real and the imaginary parts of $E(t)$ fulfill the conditions

$$E_R(t) = \lambda_t + \mathcal{O}\left(e^{-\frac{|\alpha_t|}{h} d(c, \{a, b\})}\right), \quad (2.20)$$

$$\Gamma_t = \mathcal{O}\left(e^{-\frac{|\alpha_t|}{h} d(c, \{a, b\})}\right), \quad (2.21)$$

$d(\cdot, \{a, b\})$ denoting the distance from the boundary points. The related resonant state, $G(t)$, is locally defined as the solution of

$$(H_{\theta_0, 0}^h - E(t))u = \delta_c, \quad \text{in } L^2(a, b). \quad (2.22)$$

Both $E(t)$ and $G(t)$ are holomorphic w.r.t. α , and C^∞ in time.

ii) There exists $\tau_{\chi, J} > 0$, depending on χ, J , such that the solution of (2.11)-(2.13) is

$$A_{\theta_0}(t) = a(t) + \mathcal{J}(t) + \mathcal{O}(|\theta_0|) + \tilde{\mathcal{O}}\left(e^{-\frac{\tau_{\chi, J}}{h}}\right). \quad (2.23)$$

The main contribution, $a(t)$, is described by the equation

$$\begin{cases} \partial_t a(t) = (-2\frac{\Gamma_t}{\varepsilon}) \left(a(t) - \left| \frac{\alpha_t}{\alpha_0} \right|^3 g\left(\lambda_t^{\frac{1}{2}}\right) \right) , & \text{for } d(c, \{a, b\}) = c - a , \\ a(0) = g\left(\lambda_0^{\frac{1}{2}}\right) \end{cases} \quad (2.24)$$

or by: $a(t) = \mathcal{O}\left(e^{-\frac{\beta}{h}}\right)$, $\beta = \frac{|\alpha_t|}{h}(c - a - (b - c))$ if $d(c, \{a, b\}) = b - c$.

iii) When $d(c, \{a, b\}) = c - a$, the remainder is: $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2 + \tilde{\mathcal{O}}\left(e^{-\frac{\tau_{X,K}}{h}}\right)$,

$$\mathcal{J}_1(t) = \left| 1 - \left| \frac{\alpha_t}{\alpha_0} \right|^{\frac{3}{2}} \right|^2 g\left(\lambda_t^{\frac{1}{2}}\right) = \mathcal{O}(h^2) , \quad (2.25)$$

while \mathcal{J}_2 generates a boundary layer contribution depending on the difference $\lambda_t - \lambda_0$, and whose explicit form is given by

$$\mathcal{J}_2(t) = \text{Re } 2i \left(1 - \left| \frac{\alpha_t}{\alpha_0} \right|^{\frac{3}{2}} \right) \frac{\Gamma_t}{\varepsilon} g\left(\lambda_t^{\frac{1}{2}}\right) \frac{\mathcal{T}(t)}{\frac{\lambda_t - \lambda_0}{\varepsilon} - i \frac{(\Gamma_t + \Gamma_0)}{\varepsilon}} , \quad (2.26)$$

$$\mathcal{T}(t) = \frac{|\alpha_0| \alpha_t^2 + \alpha_0^2 |\alpha_t|}{(\alpha_0 \alpha_t)^{\frac{3}{2}}} e^{-\frac{1}{\varepsilon} \int_0^t (\Gamma_\sigma + \Gamma_t) d\sigma} e^{-\frac{i}{\varepsilon} \int_0^t (\lambda_\sigma - \lambda_t) d\sigma} . \quad (2.27)$$

For $d(c, \{a, b\}) = b - c$, the correction $\mathcal{J} = \mathcal{O}\left(e^{-\frac{\beta}{h}}\right)$ is exponentially small.

The above result is concerned with situations where the two barriers composing our potential have different opacity w.r.t. the electron tunneling. When the interaction point 'c' is closer to the left boundary of the barrier, i.e. $d(c, \{a, b\}) = c - a$, a macroscopic variation of the charges accumulating around c is observed and a reduced equation is given. This corresponds to the appearance of macroscopic hysteresis phenomena in the nonlinear modelling where a similar simplified equation was predicted [18]. On the opposite, for $d(c, \{a, b\}) = b - c$, only exponentially small contributions to A_{θ_0} appears as $h \rightarrow 0$. Although the critical case, given by: $b - c = c - a + \mathcal{O}(h)$, is not explicitly considered here, most of the computations developed in this work can be adapted to study this particular problem.

The reduced model of Theorem 2.1 follows from explicit computations, which are made possible by our simplified setting. The coefficient $\left| \frac{\alpha_t}{\alpha_0} \right|^3$, appearing in this formulation, arises from the ratio of the L^2 square norms of resonant functions:

$$\frac{\|G(0)\|_{L^2(\mathbb{R})}^2}{\|G(t)\|_{L^2(\mathbb{R})}^2} .$$

This provides a possible 'link' to extend the analysis to more realistic situations. The remainders have size: $\mathcal{J}_1 = \mathcal{O}(h^2)$, while the second term can be relevant whenever $\lambda_t - \lambda_0 \sim \mathcal{O}(\varepsilon)$. Under this particular condition one has: $\mathcal{J}_2(t) = \mathcal{O}(h)$. It is pointed out in Section 5.3 that \mathcal{J}_2 coincides, out of exponentially small terms, with a scalar product between the initial state and its evolution at time t . Since we are close to the resonance, the adiabatic theorem applies, and this evolution "follows" the resonant state at time t . Thus $\mathcal{J}_2(t)$ shows a local maximum when $E(t) \sim E(0)$ and the corresponding resonant states are highly correlated.

Some of the assumptions in (h1)-(h4) can be relaxed according to the following points:

I) If we limit to the point ii) of the theorem, the condition $\alpha_t \in C^2(0, T)$ is sufficient for the derivation of the reduced model, once that a suitable adaptation of the proof of Lemma 5.2 is provided. Nevertheless, the conditions $\alpha_t \in C^\infty$ and (h2)-ii) play a central role when the asymptotics of the remainder terms is considered.

II) The relation (2.16) fixes the out-of-equilibrium condition $k > 0$ for the density matrix. The

constrain $|k^2 - \lambda_0| < 2\frac{h}{d_0}$ selects the leading term of the density kernel $\rho_t^h(\theta, x, y)$; if additional contributions to this kernel were considered, with: $|k^2 - \lambda_0| \geq 2\frac{h}{d_0}$, they would generate contributions to A_{θ_0} allowing exponentially small bounds.

The proof of Theorem 2.1 is developed in the Sections 4 and 5, and summarized at the end of Section 5.3.

3 Spectral properties of the unperturbed Hamiltonian

The spectral profile of $H_{\theta_0, \alpha(t)}^h(\theta)$, as well as the dynamics related to, are strictly connected with the Green's kernel and the generalized eigenfunctions of the corresponding unperturbed operator, $H_{\theta_0, 0}^h(\theta)$. According to the assumption (h1), in what follows we will focus our attention on the particular case: $\theta = \theta_0 = i\tau$,

$$H_{\theta_0, 0}^h(\theta_0) = -h^2 e^{-2\theta_0} 1_{\mathbb{R} \setminus (a, b)} \Delta_{2\theta_0} + 1_{(a, b)} V_0, \quad (3.1)$$

which has been considered, in a more general setting, in [11]. By making use of the results of Proposition 3.5 in [11] one has: $\sigma_{ess}(H_{\theta_0, 0}^h(\theta_0)) = e^{-2\theta_0} \mathbb{R}_+$, while a direct computation shows that the point spectrum of $H_{\theta_0, 0}^h(\theta_0)$ is formed by the solutions to the algebraic problem

$$e^{i\frac{\sqrt{z-V_0}}{h}(b-a)} = \frac{\sqrt{z-V_0} + \sqrt{z}e^{-\theta_0}}{\sqrt{z-V_0} - \sqrt{z}e^{-\theta_0}}, \quad \arg z \in \left(-\frac{3}{2}\pi, \frac{\pi}{2}\right), \quad (3.2)$$

fulfilling the condition: $\text{Im } e^{\theta_0} \sqrt{z} > 0$. These are explicitly developed in power of h as follows

$$z_n^h(\theta_0) = V_0 + h^2 \left(\frac{n\pi}{b-a} \right)^2 - 4i \frac{h^3}{(b-a)^3} (n\pi)^2 + h^4 r_{\theta_0}(n), \quad n = 0, 1, 2, \dots \quad (3.3)$$

The remainder is uniformly bounded w.r.t. θ_0 , for $|\theta_0|$ small, while: $|r_{\theta_0}(n)| = \mathcal{O}(n^2)$. In particular, for $n = 0$, the first spectral point coincide with $z_0^h(\theta_0) = V_0$.

Let $\lambda_0 \in (\delta, V_0 - \delta)$ for some $\delta > 0$, and consider the neighbourhood $\tilde{\mathcal{G}}_h(\lambda_0) \subset \subset (0, V_0)$

$$\tilde{\mathcal{G}}_h(\lambda_0) = \{z \in \mathbb{C}, |z - \lambda_0| < Ch, |\arg z| < h^{N_0}\}, \quad (3.4)$$

defined with: $\delta - Ch > 0$. For $\theta_0 = ih^{N_0}$ and $z \in \tilde{\mathcal{G}}_h(\lambda_0)$, the result of Proposition 6.5 in [11] yields the following estimates

$$\left\| (H_{\theta_0, 0}^h(\theta_0) - z)^{-1} \right\|_{\mathcal{L}(L^2(\mathbb{R}), H^1(\mathbb{R} \setminus \{a, b\}))} \leq \frac{C_{a, b, \delta}}{h^{N_0+2}}, \quad (3.5)$$

$$\left\| (H_{\theta_0, 0}^h(\theta_0) - z)^{-1} \psi \right\|_{\mathcal{L}(H^{-1}(a, b), H^1(\mathbb{R} \setminus \{a, b\}))} \leq \frac{C_{a, b, \delta, \psi}}{h^{N_0+3}}, \quad (3.6)$$

holding for any $\psi \in C_0^\infty(a, b)$, with constants depending on the data.

The generalized eigenfunctions problem for the incoming waves, in the non deformed setting $H_{\theta_0, 0}^h = -h^2 \Delta_{\theta_0} + 1_{(a, b)} V_0$, writes as

$$(H_{\theta_0, 0}^h - k^2) \tilde{\psi}_-(k, \cdot) = 0, \quad k > 0. \quad (3.7)$$

The exterior part of the solution is

$$\tilde{\psi}_-(k, \cdot) = \begin{cases} e^{i\frac{k}{h}x} + R(k)e^{-i\frac{k}{h}x}, & x < a \\ T(k)e^{i\frac{k}{h}x}, & x > b \end{cases}, \quad (3.8)$$

while, according to the boundary conditions in $D(\Delta_{\theta_0})$, the interior problem is

$$\begin{cases} (-h^2 \partial_x^2 + V_0 - k^2) \tilde{\psi}_-(k, \cdot) = 0, & \text{in } (a, b), \\ (h \partial_x + ike^{-\theta_0}) \tilde{\psi}_-(k, a^+) = 2ike^{i\frac{k}{h}a} e^{-\frac{3}{2}\theta_0}, \\ (h \partial_x - ike^{-\theta_0}) \tilde{\psi}_-(k, b^-) = 0, \end{cases} \quad (3.9)$$

It follows from a direct computation that

$$1_{(a,b)}\tilde{\psi}_-(k, \cdot) = -\frac{2 \sin \gamma_{k^2} e^{i \frac{k}{h} a} e^{-\frac{\theta_0}{2}}}{\sin \left(\frac{\Lambda_{k^2}}{h} (b-a) + 2\gamma_{k^2} \right)} \cos \left(\frac{\Lambda_{k^2}}{h} (x-b) - \gamma_{k^2} \right), \quad (3.10)$$

with

$$\Lambda_z = (z - V_0)^{\frac{1}{2}}; \quad e^{2i\gamma_z} = \frac{\Lambda_z - z^{\frac{1}{2}} e^{-\theta_0}}{\Lambda_z + z^{\frac{1}{2}} e^{-\theta_0}}. \quad (3.11)$$

The generalized eigenstates of $H_{\theta_0,0}^h(\theta_0)$ are obtained by transformation through the deformation map U_{θ_0} ; in particular, the interior part of these functions is not affected by the deformation and one has: $1_{(a,b)}U_{\theta_0}\tilde{\psi}_-(k, \cdot) = 1_{(a,b)}\tilde{\psi}_-(k, \cdot)$.

3.1 The Green's functions of $H_{\theta_0,0}^h(\theta_0)$

Assume z to be close to some limit energy λ_0 in the interval $(0, V_0)$: $z \in \tilde{\mathcal{G}}_h(\lambda_0)$. In this set, we use the square root's branch cut fixed along the positive imaginary axis (corresponding to: $\arg z \in (-\frac{3}{2}\pi, \frac{\pi}{2})$). The integral kernel of $(H_{\theta_0,0}^h(\theta_0) - z)^{-1}$ is defined by

$$(H_{\theta_0,0}^h(\theta_0) - z) G^z(\cdot, c) = \delta_c. \quad (3.12)$$

Focusing our attention on the case $c \in (a, b)$, $G^z(\cdot, c)$ writes as follows:

$$G^z(x, c)|_{\substack{x \notin (a,b) \\ c \in (a,b)}} = \begin{cases} u_+ e^{i \frac{\sqrt{z} e^{\theta_0}}{h} (x-b)}, & x > b, \\ u_- e^{-i \frac{\sqrt{z} e^{\theta_0}}{h} (x-a)}, & x < a, \end{cases} \quad (3.13)$$

while the inner problem can be rephrased as

$$\begin{cases} (-h^2 \partial_x^2 + V_0 - z) G^z(\cdot, c) = 0, & \text{for } x \in (a, b) \setminus \{c\}, \\ G^z(\cdot, c) \in H^1(a, b); \quad h^2 (\partial_x G^z(c^+, c) - \partial_x G^z(c^-, c)) = -1, \\ (h \partial_x + i \sqrt{z} e^{-\theta_0}) G^z(a^+, c) = 0, \\ (h \partial_x - i \sqrt{z} e^{-\theta_0}) G^z(b^-, c) = 0, \end{cases} \quad (3.14)$$

Whit the notation adopted in (3.11), the solution is

$$G^z(x, c) = -\frac{1}{h \Lambda_z} \frac{1}{\tan \left(\frac{\Lambda_z}{h} (c-a) + \gamma_z \right) - \tan \left(\frac{\Lambda_z}{h} (c-b) - \gamma_z \right)} \cdot \begin{cases} \frac{\cos \left(\frac{\Lambda_z}{h} (x-a) + \gamma_z \right)}{\cos \left(\frac{\Lambda_z}{h} (c-a) + \gamma_z \right)}, & x \in (a, c), \\ \frac{\cos \left(\frac{\Lambda_z}{h} (x-b) - \gamma_z \right)}{\cos \left(\frac{\Lambda_z}{h} (c-b) - \gamma_z \right)}, & x \in (c, b). \end{cases} \quad (3.15)$$

It follows from the definition of $\tilde{\mathcal{G}}_h(\lambda_0)$ and our choice of the branch cut, that: $\text{Im} \sqrt{z} e^{\theta_0} > 0$. Thus (3.13) and (3.15), properly defines L^2 -functions. When $\text{Re } z \in (0, V_0)$ – as is the case for $z \in \tilde{\mathcal{G}}_h(\lambda_0)$ – the point $z - V_0$ has a negative real part and, according to the definition of the square root, one has: $\text{Im } \Lambda_z < 0$. Therefore, the terms of the type: $e^{i \frac{\Lambda_z}{h} d}$ appearing in (3.15) are exponentially increasing or decreasing functions of $\frac{1}{h}$ depending on the sign of d . The asymptotic behaviour of the value $G^z(c, c)$ as $h \rightarrow 0$ will be considered by using the formula: $l = (b-a)$, $p_{\theta_0}(z) = e^{-2i\gamma_z}$

$$G^z(c, c) = -\frac{i}{2h \Lambda_z} \frac{1}{1 - e^{-2i \frac{\Lambda_z}{h} l} p_{\theta_0}^2(z)} \left[1 + e^{-2i \frac{\Lambda_z}{h} (c-a)} p_{\theta_0}(z) + e^{-2i \frac{\Lambda_z}{h} (b-c)} p_{\theta_0}(z) + e^{-2i \frac{\Lambda_z}{h} l} p_{\theta_0}^2(z) \right] \quad (3.16)$$

which is a rewriting of (3.15).

Remark 3.1 Although G^z is properly defined for $z \in \mathbb{C} \setminus \mathbb{R}_+ e^{-2\theta_0} \cup \{z_n^h(\theta_0)\}$, and in particular for $z \in \tilde{\mathcal{G}}_h(\lambda_0)$, the relation (3.16) makes sense in $\mathbb{C} \setminus \{z_n^h(\theta_0)\}$. Thus, considering the small- h expansions of $G^z(c, c)$, a larger neighbourhood of λ_0 can be used, such that: $\text{Im } \Lambda_z < 0$.

Next we give accurate upper and lower bounds and exponential estimates, for $G^z(\cdot, c)$, $c \in (a, b)$. Using (3.13)-(3.15), allows to show that this function is exponentially decaying outside a small neighbourhood of $x = c$, where all its mass concentrates as $h \rightarrow 0$.

Lemma 3.2 Let $z \in \tilde{\mathcal{G}}_h(\lambda_0)$, $\theta_0 = ih^{N_0}$ and $h \in (0, h_0)$ with h_0 small. For $c \in (a, b)$, the following estimates holds

$$\frac{c_0}{h} \leq \|G^z(\cdot, c)\|_{L^2(a,b)}^2 \leq \frac{c_1}{h}, \quad (3.17)$$

$$\sup_{[a,b]} \left| e^{\frac{\varphi}{h}} G^z(\cdot, c) \right| + \left\| e^{\frac{\varphi}{h}} h \partial_x G^z(\cdot, c) \right\|_{L^2(a,b)} + \left\| e^{\frac{\varphi}{h}} G^z(\cdot, c) \right\|_{L^2(a,b)} \leq \frac{C_{a,b}}{h}, \quad (3.18)$$

$$\|G^z(\cdot, c)\|_{L^2(\mathbb{R} \setminus (a,b))} \leq C_{a,b} h^{-\frac{N_0+1}{2}} e^{-\frac{\beta(\lambda_0)}{h}}, \quad (3.19)$$

$$\|G^{z_1} - G^{z_2}\|_{L^2(\mathbb{R})} \leq \frac{C_{a,b,\delta}}{h^{N_0+2}} |z_1 - z_2|; \quad \|\partial_z G^z(\cdot, c)\|_{L^2(\mathbb{R})} \leq \frac{C_{a,b,c,\delta}}{h^{N_1}}, \quad (3.20)$$

with: $\varphi = (V_0 - \lambda_0)^{\frac{1}{2}} |\cdot - c|$, $\beta(\lambda_0) = (V_0 - \lambda_0)^{\frac{1}{2}} d(c, \{a, b\})$, $N_1 = 2N_0 + 5$ and constants depending on the data.

Proof. To simplify the notation, we use G^z instead of $G^z(\cdot, c)$. Let start considering an upper bound of $\|G^z\|_{L^2(a,b)}$ in the relevant energy range: $\text{Re } z \in (0, V_0)$. Owing to (3.15), we have

$$G^z|_{x \in (a,b) \setminus \{c\}} = \frac{1}{ih\Lambda_z} e^{-i\frac{\Lambda_z}{h}|x-c|} \left[1 + \mathcal{O}\left(e^{-2i\frac{\Lambda_z}{h}d(x, \{a,b\})}\right) \right].$$

Since $\text{Im } \Lambda_z < 0$ when $\text{Re } z \in (0, V_0)$, the quantities: $e^{-i\frac{\Lambda_z}{h}|x-c|}$, $e^{-2i\frac{\Lambda_z}{h}d(x, \{a,b\})}$ are exponentially small as $h \rightarrow 0$, and an explicit computation yields

$$\|G^z(\cdot, c)\|_{L^2(a,b)}^2 \leq \frac{C_1}{h |\text{Im } \Lambda_z|} \leq \frac{\tilde{C}_1}{h}. \quad (3.21)$$

For the lower bound, we refer to (3.15), with $x \in (a, c)$, to get

$$G^z(x)|_{x \in (a,c)} \geq \frac{C_0}{h} \left| \frac{\cos\left(\frac{\Lambda_z}{h}(x-a) + \gamma_z\right)}{\cos\left(\frac{\Lambda_z}{h}(c-a) + \gamma_z\right)} \right| \geq \frac{\tilde{C}_0}{h} \left| e^{-i\frac{\Lambda_z}{h}(c-x)} \right|,$$

for δ and h small. It follows

$$\|G^z\|_{L^2(a,b)}^2 \geq \|G^z\|_{L^2(c-\delta,c)}^2 \geq \frac{\tilde{C}_0}{h^2} \int_{c-\delta}^c \left| e^{-i\frac{\Lambda_z}{h}(c-x)} \right|^2 dx \geq \frac{\tilde{C}_0}{4h |\text{Im } \Lambda_z|}. \quad (3.22)$$

Exponential estimates are usually obtained from Agmon identities using as exponential weight the distance from the classical region of motion (we refer to [12]). However, in this particular case, the explicit formula (3.15) shows that the leading factors in G^z and $h\partial_x G^z$ are controlled by:

$e^{-i\frac{\Lambda_z}{h}|x-c|} \lesssim e^{-\frac{(V_0-\lambda_0)^{\frac{1}{2}}}{h}|x-c|}$; this gives (3.18). In the exterior domain, a direct computation yields

$$\|G^z\|_{L^2(\mathbb{R} \setminus (a,b))}^2 = \frac{h}{2 \text{Im}(ze^{2\theta_0})^{\frac{1}{2}}} \left(|G^z(a^+)|^2 + |G^z(b^-)|^2 \right). \quad (3.23)$$

For $z \in \tilde{\mathcal{G}}_h(\lambda_0)$, $\text{Im}(ze^{2\theta_0})^{\frac{1}{2}} \sim \mathcal{O}(h^{N_0})$, and we get

$$\|G^z\|_{L^2(\mathbb{R} \setminus (a,b))}^2 \leq \frac{1}{2h^{N_0-1}} \left(|G^z(a^+)|^2 + |G^z(b^-)|^2 \right). \quad (3.24)$$

According to (3.18), the boundary values of G^z are estimated by $\frac{e^{-\frac{\beta(\lambda_0)}{h}}}{h}$ and the inequality (3.19) follows.

From the definition (3.13)-(3.15), $z \rightarrow G^z$ is an L^2 -valued holomorphic map in $\tilde{\mathcal{G}}_h(\lambda_0)$. For $z_1, z_2 \in \tilde{\mathcal{G}}_h(\lambda_0)$, we have

$$\begin{aligned} G^{z_1} - G^{z_2} &= \left[(H_{\theta_0,0}^h(\theta_0) - z_1)^{-1} - (H_{\theta_0,0}^h(\theta_0) - z_2)^{-1} \right] \delta_c \\ &= (z_1 - z_2) (H_{\theta_0,0}^h(\theta_0) - z_1)^{-1} (H_{\theta_0,0}^h(\theta_0) - z_2)^{-1} \delta_c = (z_1 - z_2) (H_{\theta_0,0}^h(\theta_0) - z_1)^{-1} G^{z_2}. \end{aligned} \quad (3.25)$$

Using (3.5), the first of (3.20) follows. The derivative $\partial_z G^z$, is expressed by

$$\partial_z G^z = (H_{\theta_0,0}^h(\theta_0) - z)^{-1} G^z, \quad (3.26)$$

which implies: $\partial_z G^z = (H_{\theta_0,0}^h(\theta_0) - z)^{-2} \delta_c$. Then, using (3.5)-(3.6) completes the proof of (3.20). \blacksquare

4 A Krein's resolvent formula and spectral expansions for small h

Let consider the spectral problem for the deformed Hamiltonian with $\theta = \theta_0 = ih^{N_0}$,

$$H_{\theta_0,\alpha}^h(\theta_0) = -h^2 e^{-2\theta_0} 1_{\mathbb{R} \setminus (a,b)} \Delta_{2\theta_0} + 1_{(a,b)} V_0 + h\alpha \delta_c, \quad c \in (a,b). \quad (4.1)$$

The related resolvent operator can be expressed as a finite rank perturbation of $(H_{\theta_0,0}^h(\theta_0) - z)^{-1}$

$$(H_{\theta_0,\alpha}^h(\theta_0) - z)^{-1} = (H_{\theta_0,0}^h(\theta_0) - z)^{-1} - \frac{h\alpha \langle \bar{G}^z(\cdot, c), \cdot \rangle_{L^2(R)}}{1 + h\alpha G^z(c, c)} G^z(\cdot, c). \quad (4.2)$$

This will provide an accurate description of the resonant energy as $h \rightarrow 0$.

Proposition 4.1 *Let $h \in (0, h_0)$, with h_0 small, $\theta_0 = ih^{N_0}$ and $\alpha \in (-2V_0^{\frac{1}{2}}, 0)$. The spectrum of $H_{\theta_0,\alpha}^h(\theta_0)$ is characterized by the conditions:*

i) *The essential spectrum is: $\sigma_{ess}(H_{\theta_0,\alpha}^h(\theta_0)) = e^{-2\theta_0} \mathbb{R}_+$.*

ii) *There exists a unique nondegenerate spectral point of $H_{\theta_0,\alpha}^h(\theta_0)$ in $\{\operatorname{Re} z \in (0, V_0), \arg z \in (-2\theta_0, 0)\}$, admitting the small- h expansion*

$$E_{res}^h = V_0 - \frac{\alpha^2}{4} - \frac{\alpha^2}{2} p_0(E^0) e^{-\frac{|\alpha|}{h} d(c, \{a,b\})} + \mathcal{O}\left(\theta_0 e^{-\frac{|\alpha|}{h} d(c, \{a,b\})}\right) + o(e^{-\frac{|\alpha|}{h} d(c, \{a,b\})}), \quad (4.3)$$

with: $p_0(E^0) = \frac{1}{V_0} \left[i|\alpha| \left(V_0 - \frac{\alpha^2}{4} \right)^{\frac{1}{2}} - \left(V_0 - \frac{\alpha^2}{2} \right) \right]$. The corresponding eigenvector is given by the Green's function: $G^{E_{res}^h}(\cdot, c)$.

iii) *Both E_{res}^h and $G^{E_{res}^h}(\cdot, c)$ are holomorphic w.r.t. α .*

Proof. i) The first statement is a consequence of Corollary 3.4 in [11] (holding for generic \mathcal{M}_b -perturbation of $H_{\theta_0,0}^h(\theta_0)$ supported in (a,b)).

ii) According to Proposition 5.5 in [11] (partly relying on Helffer-Sjöstrand techniques in [13]), the points in $\{\operatorname{Re} z \in (0, V_0), \arg z \in (-2\theta_0, 0)\} \cap \sigma(H_{\theta_0,0}^h(\theta_0))$ are localized around the eigenvalues of the Dirichlet Hamiltonian: $H_D^h = -\Delta_{(a,b)}^D + 1_{(a,b)} V_0 + \alpha \delta_c$, with a one to one correspondence and an exponentially small bound. Using the dilation: $x \rightarrow \frac{x-c}{h}$, we can refer to the spectral problem

for $H_d^h = -\Delta_{\left(\frac{\bar{a}}{h}, \frac{\bar{b}}{h}\right)} + 1_{\left(\frac{\bar{a}}{h}, \frac{\bar{b}}{h}\right)} V_0 + \alpha \delta_c$. When $h \rightarrow 0$, the spectral subset $(0, V_0) \cap \sigma(H_d^h)$ converges to $(0, V_0) \cap \sigma(H_d^0)$, with: $H^0 = -\Delta + V_0 + \alpha \delta_c$, preserving the dimension of the respective subspaces (the proof of this point is based on standard convergence estimates in semiclassical analysis; a guide line for it can be recovered from the strategy used in [10] Lemma 4.5). The point spectrum of H^0 is explicitly computable: for $\alpha \in (-2V_0^{\frac{1}{2}}, 0)$, it is composed of a unique point of multiplicity 1 and equal to $\lambda = V_0 - \frac{|\alpha|^2}{4}$. Therefore, there exists a unique non-degenerate eigenvalue, E_{res}^h , of $H_{\theta_0, \alpha}^h(\theta_0)$ in the prescribed region, converging to λ as $h \rightarrow 0$. The result of Proposition 5.5 in [11], writes in this case as: $E_{res}^h = V_0 - \frac{\alpha^2}{4} + \mathcal{O}(h^{-3} e^{-\frac{|\alpha|}{h} d(c, \{a, b\})})$.

A more refined asymptotic expression is obtained by using the explicit resolvent's formula. Since the poles of $\left(H_{\theta_0, 0}^h(\theta_0) - z\right)^{-1}$ are confined in $\{\operatorname{Re} z \geq V_0\}$ (we refer to (3.3)), the relation (4.2) leads to an equation for E_{res}^h

$$1 + h\alpha G^E(c, c) = 0 \quad (4.4)$$

In the strip $\operatorname{Re} E \in (0, V_0)$, where: $\operatorname{Im}(E - V_0)^{\frac{1}{2}} < 0$ due to the determination $\arg z \in (-\frac{3}{2}\pi, \frac{\pi}{2})$, the above equation is rephrased as

$$2(E - V_0)^{\frac{1}{2}} - \frac{i\alpha}{1 - e^{-2i\frac{(E-V_0)^{\frac{1}{2}}}{h}} l} \left[1 + e^{-2i\frac{(E-V_0)^{\frac{1}{2}}}{h}(c-a)} p_{\theta_0}(E) + e^{-2i\frac{(E-V_0)^{\frac{1}{2}}}{h}(b-c)} p_{\theta_0}(E) + e^{-2i\frac{(E-V_0)^{\frac{1}{2}}}{h}} l p_{\theta_0}^2(E) \right] = 0, \quad (4.5)$$

according to (3.16). Let $E = V_0 - \frac{\alpha^2}{4} + \delta E e^{-\frac{|\alpha|}{h} d(c, \{a, b\})} + o(e^{-\frac{|\alpha|}{h} d(c, \{a, b\})})$; an approximation of the first order in $e^{-\frac{|\alpha|}{h} d(c, \{a, b\})}$ of (4.5) yields

$$\delta E = -\frac{\alpha^2}{2} p_{\theta_0}(E^0), \quad (4.6)$$

where $p_{\theta_0}(E)$ is holomorphic w.r.t. both the variables provided that $E \sim V_0 - \frac{\alpha^2}{4}$ and $|\theta_0| \ll 1$. The value $p_{\theta_0}(E^0)$ is approximated by: $p_{\theta_0}(E^0) = \gamma_0(E^0) + \mathcal{O}(\theta_0)$,

$$p_0(E^0) = \frac{1}{V_0} \left[i|\alpha| \left(V_0 - \frac{\alpha^2}{4} \right)^{\frac{1}{2}} - \left(V_0 - \frac{\alpha^2}{2} \right) \right],$$

this leads to the expansion (4.3). Finally, the relation (4.4), allows to verify that:

(iii) Consider the quadratic form associated with the operator $H_{\theta_0, \alpha}^h(\theta_0)$. This is an accretive form (due to the choice $\theta = \theta_0$) and its domain, $Q(H_{\theta_0, 0}^h(\theta_0)) = H^2(\mathbb{R} \setminus \{a, c, b\}) \cup H^1(\mathbb{R} \setminus \{a, b\})$, is independent of h and α . For any $u \in Q(H_{\theta_0, 0}^h(\theta_0))$, its action

$$\langle u, H_{\theta_0, \alpha}^h(\theta_0) u \rangle_{L^2(\mathbb{R})} = h^2 \sin 2\theta_0 \int_{\mathbb{R} \setminus (a, b)} |u'|^2 + V_0 \int_{\mathbb{R}} |u|^2 + h\alpha |u(c)|^2$$

defines an holomorphic function of α . Thus, $H_{\theta_0, \alpha}^h(\theta_0)$ is an analytic family type B w.r.t. α and the Kato-Rellich Theorem applies to the non-degenerate discrete eigenvalue E_{res}^h . ■

Remark 4.2 The solution E_{res}^h is a singularity of the resolvent embedded in the second Riemann sheet and corresponds to the shape-resonance produced by the attractive part of the interaction. It defines an eigenvalue of the deformed operator $H_{\theta_0, \alpha}^h(\theta_0)$ provided that $|\arg E_{res}^h|$ is lower than the deformation angle, given by h^{N_0} in our assumption. Since $\operatorname{Im} E_{res}^h$ is exponentially small w.r.t. h this condition definitively holds as $h \rightarrow 0$.

Next, we consider the expansions of relevant quantities involved in the computation of $A_{\theta_0}(t)$ for energies close to the resonance. In what follows, we assume the results of Proposition 4.1 to hold, and take $|E - E_{res}^h| < \frac{h}{d}$, d being a small constant which fixes a complex neighbourhood of E_{res}^0 of size h . In such a domain, the relation (3.16) can be used to write expansions of $G^E(c, c)$ as $h \rightarrow 0$ (see Remark 3.1). Recalling that: $1 + h\alpha G^{E_{res}^h}(c, c) = 0$, the function $(1 + h\alpha G^E(c, c))^{-1}$ can be written in the form

$$(1 + h\alpha G^E(c, c))^{-1} = \frac{M(E, E_{res}^h)}{E - E_{res}^h}, \quad (4.7)$$

$$M(E, E_{res}^h) = \frac{E - V_0 + (E_{res}^h - V_0)^{\frac{1}{2}} (E - V_0)^{\frac{1}{2}}}{1 + h\alpha \left[(E - V_0)^{\frac{1}{2}} + (E_{res}^h - V_0)^{\frac{1}{2}} \right] \frac{(E - V_0)^{\frac{1}{2}} G^E(c, c) - (E_{res}^h - V_0)^{\frac{1}{2}} G^{E_{res}^h}(c, c)}{E - E_{res}^h}}, \quad (4.8)$$

with the branch-cut fixed along $i\mathbb{R}_+$. The incremental ratio at the denominator in (4.8) is controlled by the derivative of $(E - V_0)^{\frac{1}{2}} G^E(c)$ evaluated in a neighbourhood of E_{res}^h . According to (3.16), this writes as

$$(E - V_0)^{\frac{1}{2}} G^E(c, c) = -\frac{i}{2h} \frac{1}{1 - e^{-2i\frac{(E-V_0)^{\frac{1}{2}}}{h}l} p_{\theta_0}^2(E)} \left[1 + e^{-2i\frac{(E-V_0)^{\frac{1}{2}}}{h}(c-a)} p_{\theta_0}(E) + e^{-2i\frac{(E-V_0)^{\frac{1}{2}}}{h}(b-c)} p_{\theta_0}(E) + e^{-2i\frac{(E-V_0)^{\frac{1}{2}}}{h}l} p_{\theta_0}^2(E) \right].$$

Using the holomorphicity of $p_{\theta_0}(E)$ and $(E - V_0)^{\frac{1}{2}}$ in $\{|E - E_{res}^h| < \frac{h}{d}\}$, and the asymptotic characterization (4.3), we get:

$$\partial_E (E - V_0)^{\frac{1}{2}} G^E(c, c) = h^{-2} e^{-\frac{|\alpha|}{h} d(c, \{a, b\})} \mathcal{R}(E). \quad (4.9)$$

This yields the representation

$$M(E, E_{res}^h) = \left[E - V_0 + (E_{res}^h - V_0)^{\frac{1}{2}} (E - V_0)^{\frac{1}{2}} \right] + h^{-1} e^{-\frac{|\alpha|}{h} d(c, \{a, b\})} \mathcal{R}(E), \quad (4.10)$$

holding for $|E - E_{res}^h| < \frac{h}{d}$. In a closer neighbourhood of the resonance, the function $M(E, E_{res}^h)$ is connected with scalar products of Green's functions.

Lemma 4.3 *In the assumptions of Proposition 4.1, let $E \in \tilde{\mathcal{G}}_h(E_{res}^h)$, $c \in (a, b)$, $S = |\alpha| d(c, \{a, b\})$. The relations*

$$h\alpha \left\langle G^{E^*}(\cdot, c), G^{E_{res}^h}(\cdot, c) \right\rangle_{L^2(\mathbb{R})} = \frac{1}{M(E, E_{res}^h)} + h^{-N} e^{-\frac{S}{h}} \mathcal{R}_0(E, h), \quad (4.11)$$

$$h\alpha \left\langle G^{E^*}(\cdot, c), \partial_z G^z(\cdot, c) \right\rangle_{L^2(\mathbb{R})} \Big|_{z=E_{res}^h} = \partial_z \frac{1}{M(E, z)} \Big|_{z=E_{res}^h} + h^{-N_1} e^{-\frac{S}{h}} \mathcal{R}_1(E, h), \quad (4.12)$$

hold with: N, N_1 suitable positive integers, while $\mathcal{R}_i(E, h)$ are holomorphic functions of E uniformly bounded w.r.t. h .

Proof. From the relations $h\alpha G^{E_{res}^h}(c, c) = -1$ and (3.25), it follows

$$(1 + h\alpha G^E(c, c)) = h\alpha \left(G^E(c, c) - G^{E_{res}^h}(c, c) \right) = h\alpha (E - E_{res}^h) (H_{\theta_0, 0}^h(\theta_0) - E)^{-1} G^{E_{res}^h}(c, c), \quad (4.13)$$

which also writes as

$$(1 + h\alpha G^E(c, c)) = h\alpha (E - E_{res}^h) \int_{\mathbb{R}} G^E(c, x) G^{E_{res}^h}(x, c) dx. \quad (4.14)$$

For $x \in (a, b)$, $G^E(c, x)$ is obtained from (3.15) by interchanging the variables x and c ; from a direct check on this formula it follows: $G^E(x, c) = G^E(c, x)$. For any $x \in \mathbb{R} \setminus (a, b)$, the map $c \rightarrow G^E(c, x)$ is the solution of: $(H_{\theta_0, 0}^h(\theta_0) - z) G^z(\cdot, x) = \delta_x$ in (a, b) . According to the boundary conditions in (2.2), this problem explicitly writes as

$$\begin{cases} (-h^2 \partial_c^2 + V_0 - E) G^E(c, x) = 0 & \text{for } c \in (a, b), x > b \quad \text{for } c \in (a, b), x < a, \\ \left(h \partial_c + i \sqrt{E} e^{-\theta_0} \right) G^E(a^+, x) = & 0 & -\frac{e^{-\theta_0}}{h} e^{i \frac{\sqrt{E} e^{\theta_0}}{h} (a-x)}, \\ \left(h \partial_c - i \sqrt{E} e^{-\theta_0} \right) G^E(b^-, x) = & \frac{e^{-\theta_0}}{h} e^{i \frac{\sqrt{E} e^{\theta_0}}{h} (x-b)} & 0. \end{cases} \quad (4.15)$$

For $E \in \tilde{\mathcal{G}}_h(E_{res}^h)$, the Lemma 4.3 in [11] applies, and a point-wise exponential estimate for $1_{(a,b)} G^E(\cdot, x)$ holds, depending on x :

$$\sup_{c \in [a,b]} \left| e^{\frac{\varphi}{h}} G^E(c, x) \right| \leq \frac{C_{a,b}}{h^2} \left(\left| 1_{x>b} e^{i \frac{\sqrt{E} e^{\theta_0}}{h} (x-b)} \right| + \left| 1_{x<a} e^{i \frac{\sqrt{E} e^{\theta_0}}{h} (a-x)} \right| \right), \quad (4.16)$$

with: $\varphi = \frac{|\alpha|}{2} d(\cdot, \{a, b\})$. From $i)$, the integral at the r.h.s. of (4.14) writes as

$$\begin{aligned} \int_{\mathbb{R}} G^E(c, x) G^{E_{res}^h}(x, c) dx &= \int_a^b G^E(x, c) G^{E_{res}^h}(x, c) dx + \int_{\mathbb{R} \setminus (a,b)} G^E(c, x) G^{E_{res}^h}(x, c) dx \\ &= \int_{\mathbb{R}} G^E(x, c) G^{E_{res}^h}(x, c) dx - \int_{\mathbb{R} \setminus (a,b)} [G^E(x, c) - G^E(c, x)] G^{E_{res}^h}(x, c) dx. \end{aligned}$$

The exterior contribution, defines an holomorphic function of $E \in \tilde{\mathcal{G}}_h(E_{res}^h)$. From (4.16) the Cauchy-Schwarz inequality and the estimate (3.19), applied with $\lambda_0 = V_0 - \frac{\alpha^2}{4}$, this is bounded by $\mathcal{O}\left(h^{-N} e^{-\frac{\varphi}{h}}\right)$ for a suitable large N , uniformly w.r.t. E . Then (4.11) is deduced from (4.7). The second relation (4.12) similarly follows by using (3.20). ■

5 Adiabatic evolution of $A_{\theta_0}(t)$.

We consider the asymptotic behaviour of the dynamical system (2.11)-(2.13) as $h \rightarrow 0$ goes to zero. The assumptions (h1)-(h4) fix the physical data of the problem, including: 1) the quantum observable χ , corresponding to the charge accumulating around the interaction point c ; 2) the time-profile of the interaction, $\alpha(t)$, which determines the resonant energy level at time t ; 3) the energy partition function g , defining an out-of-equilibrium initial state; 4) the long time scale of the problem, corresponding to the inverse of the adiabatic parameter ε defined in (2.18).

In particular, the constraint $\alpha_t \in (-2V_0^{\frac{1}{2}}, 0)$ implies that the attractive part of the interaction generates, for each t , a single resonance whose small- h expansion is given in (4.3). With the notation introduced in Section 2, this corresponds to: $E(t) = E_R(t) - i\Gamma_t$

$$E_R(t) = V_0 - \frac{\alpha_t^2}{4} + \mathcal{O}\left(e^{-\frac{|\alpha_t|}{h} d(c, \{a, b\})}\right) \quad (5.1)$$

$$\Gamma_t = \mathcal{O}\left(e^{-\frac{|\alpha_t|}{h} d(c, \{a, b\})}\right) \quad (5.2)$$

Due to (2.15), exponentially small terms $\mathcal{O}\left(e^{-\frac{|\alpha_t|}{h} d(c, \{a, b\})}\right)$ can be replaced by $\mathcal{O}\left(e^{-\frac{|\alpha_0|}{h} d(c, \{a, b\})}\right)$; this leads to

$$\Gamma_t = \mathcal{O}\left(e^{-\frac{|\alpha_0|}{h} d(c, \{a, b\})}\right) = \mathcal{O}(\varepsilon) \quad \forall t \quad (5.3)$$

where the definition (2.18) is taken into account. The corresponding resonant state, given by the Green's function $G^{E(t)}(\cdot, c)$, will be simply denoted by $G(t)$.

The condition $\theta_0 = ih^{N_0}$ in (h1) allows to control the perturbation introduced by the interface conditions: namely, the distance between $E(t)$ and the corresponding resonant level for the unperturbed model is bounded by: $\mathcal{O}(\theta_0 e^{-\frac{|\alpha_t|}{h} d(c, \{a, b\})})$, according to (4.3). A suitable choice of the parameter d_0 in (h2)-(h3), ensures that: $E_R^{\frac{1}{2}}(t) \subset \text{supp } g \subset (0, V_0)$ definitely holds as $h \rightarrow 0$.

The conditions (h1)-(h4) also provide with a well-posed functional analytical framework for the study of the adiabatic problem. According to the result of Proposition 3.7(d) in [11], the Hamiltonian $-\frac{i}{\varepsilon} H_{\theta_0, \alpha(t)}^h(\theta_0)$, with $\alpha_t \in C^2((0, T), \mathbb{R})$, generates a dynamical system of contractions, $S^\varepsilon(t, s)$, $t \geq s$, defined by the equation

$$i\varepsilon \partial_t S^\varepsilon(t, s) = H_{\theta_0, \alpha(t)}^h(\theta_0) S^\varepsilon(t, s), \quad S^\varepsilon(s, s) = Id, \quad (5.4)$$

which preserves the domains, $S^\varepsilon(t, s) D(H_{\theta_0, \alpha(s)}^h) \subset D(H_{\theta_0, \alpha(t)}^h)$ for $t \geq s$. An adiabatic theorem, for arbitrarily large time scales $\varepsilon = e^{-\frac{t}{h}}$, has been proved to hold for a wide class of Hamiltonians with interface conditions and exterior complex dilations, including the case of $H_{\theta_0, \alpha(t)}^h(\theta_0)$ (Theorem 7.1 in [11]). To fix this point, consider the adiabatic evolution of the initial resonant state $G(0)$; our problem is

$$i\varepsilon \partial_t u = H_{\theta_0, \alpha(t)}^h(\theta_0) u, \quad u_{t=0} = G(0), \quad (5.5)$$

where ε is fixed to the exponentially small scale $\varepsilon = e^{-\frac{|\alpha_0|}{h} d(c, \{a, b\})}$. Let $\mathcal{G}_h(E(t))$ denotes the set

$$\mathcal{G}_h(E(t)) = \tilde{\mathcal{G}}_h(E(t)) \setminus \left\{ z \in \mathbb{C}, d(z, E(t)) \geq \frac{h^{N_0}}{C} \right\}$$

with $\tilde{\mathcal{G}}_h(\cdot)$ defined by (3.4). For C suitably large, this forms a non-empty subset of $\mathcal{G}_h(E(t))$ where we can define the normalized non-selfadjoint projector on $G(t)$ as

$$P(t) = \frac{1}{2\pi i} \int_{\gamma^h(t)} \left(z - H_{\theta_0, \alpha(t)}^h(\theta_0) \right)^{-1} dz \quad (5.6)$$

being $\gamma^h(t)$ a smooth curve in $\mathcal{G}_h(E(t))$ simply connected to $E(t)$. With this notation, the result of [11] rephrases as follows

$$\sup_{t \in [0, T]} |S^\varepsilon(t, s) G(0) - \phi_t|_{L^2(\mathbb{R})} \leq \tilde{\mathcal{O}}(\varepsilon) |G(0)|_{L^2(\mathbb{R})} \quad (5.7)$$

$$\phi_t = \mu(t) e^{-\frac{i}{\varepsilon} \int_0^t E(s) ds} G(t), \quad \text{and: } \begin{cases} \dot{\mu}(t) |G(t)|_{L^2(\mathbb{R})}^2 = -\mu(t) \langle G(t), P(t) \partial_t G(t) \rangle \\ \mu(0) = 1 \end{cases} \quad (5.8)$$

The coefficient at the r.h.s. of the equation can be made explicit according to $P(t) = \frac{\langle G^*(t), \cdot \rangle G(t)}{\langle G^*(t), G(t) \rangle}$, where $G^*(t) = G^{E^*(t)}$ is the anti-resonant function. Using the inequalities (3.17) and (3.20), we have: $G(t) - G^*(t) = \tilde{\mathcal{O}}(E(t) - E^*(t)) = \tilde{\mathcal{O}}(\varepsilon)$ in L^2 , and

$$\begin{aligned} \langle G(t), P(t) \partial_t G(t) \rangle_{L^2(\mathbb{R})} &= \frac{|G(t)|_{L^2(\mathbb{R})}^2}{\langle G^*(t), G(t) \rangle_{L^2(\mathbb{R})}} \langle G^*(t), \partial_t G(t) \rangle_{L^2(\mathbb{R})} \\ &= \frac{|G(t)|_{L^2(\mathbb{R})}^2}{|G(t)|_{L^2(\mathbb{R})}^2 + \tilde{\mathcal{O}}(\varepsilon)} \left(\langle G(t), \partial_t G(t) \rangle_{L^2(\mathbb{R})} + \tilde{\mathcal{O}}(\varepsilon) \right) = \langle G(t), \partial_t G(t) \rangle_{L^2(\mathbb{R})} + \tilde{\mathcal{O}}(\varepsilon) \end{aligned}$$

This provides with an expansion for $\mu(t)$

$$\mu(t) = e^{-\int_0^t \frac{\langle G(s), \partial_s G(s) \rangle}{|G(s)|_2^2} ds} + \tilde{\mathcal{O}}(\varepsilon) \quad (5.9)$$

5.1 A decomposition of $A_{\theta_0}(t)$

We shall use a decomposition in the same spirit of the one proposed in [14] and [18] with additional specific information given by our specific model. For $\theta = \theta_0$, the Cauchy problem (2.13) writes as

$$\begin{cases} i\varepsilon \partial_t u_{\theta_0}(k, \cdot, t) = H_{\theta_0, \alpha(t)}^h(\theta_0) u_{\theta_0}(k, \cdot, t), \\ u_{t=0} = U_{\theta_0} \psi_-(k, \cdot, \alpha_0), \end{cases} \quad (5.10)$$

where, for a fixed α , $U_{\theta_0} \psi_-(k, \cdot, \alpha)$ solves the equation

$$(H_{\theta_0, \alpha}^h(\theta_0) - k^2) U_{\theta_0} \psi_-(k, \cdot, \alpha) = 0, \quad H_{\theta_0, \alpha}^h(\theta_0) u \in L_{loc}^2. \quad (5.11)$$

For time dependent α , the following representation holds

$$U_{\theta_0} \psi_-(k, \cdot, \alpha(t)) = U_{\theta_0} \tilde{\psi}_-(k, \cdot) + C(k, t) G^{k^2}, \quad (5.12)$$

where $\tilde{\psi}_-(k, \cdot)$ are the incoming scattering states of the unperturbed Hamiltonian (solving (3.7)), the coefficient $C(k, t)$ is defined according to

$$C(k, t) = -\frac{h\alpha_t \tilde{\psi}_-(k, c)}{1 + h\alpha_t G^{k^2}(c)}, \quad (5.13)$$

while G^{k^2} is explicately given in (3.13)-(3.15). A possible decomposition of the solution of (5.10) is

$$u_{\theta_0}(k, \cdot, t) = e^{-i\frac{t}{\varepsilon}k^2} U_{\theta_0} \psi_-(k, \cdot, \alpha(t)) + R(t). \quad (5.14)$$

Denoting $\varphi_t = e^{-i\frac{t}{\varepsilon}k^2} U_{\theta_0} \psi_-(k, \cdot, \alpha(t))$, we have

$$\begin{cases} i\varepsilon \partial_t \varphi_t = k^2 \varphi_t + i\varepsilon e^{-i\frac{t}{\varepsilon}k^2} \dot{C}(k, t) G^{k^2} = H_{\theta_0, \alpha(t)}^h(\theta_0) \varphi_t + i\varepsilon e^{-i\frac{t}{\varepsilon}k^2} \dot{C}(k, t) G^{k^2}, \\ \varphi_0 = U_{\theta_0} \psi_-(k, \cdot, \alpha_0). \end{cases} \quad (5.15)$$

As it follows from (2.13) and (5.15), the remainder $R(t) = u_{\theta_0}(k, \cdot, t) - \varphi_t$ solves the Cauchy problem

$$\begin{cases} i\varepsilon \partial_t R(t) = H_{\theta_0, \alpha(t)}^h(\theta_0) R(t) + i\varepsilon e^{-i\frac{t}{\varepsilon}k^2} \dot{C}(k, t) G^{k^2}, \\ R(0) = 0, \end{cases} \quad (5.16)$$

and its explicit form is

$$R(t) = -\int_0^t S^\varepsilon(t, s) e^{-i\frac{s}{\varepsilon}k^2} \dot{C}(k, s) G^{k^2} ds, \quad (5.17)$$

where $S^\varepsilon(t, s)$ is the dynamical system associated with $-\frac{i}{\varepsilon} H_{\theta_0, \alpha(t)}^h(\theta_0)$. Making use of (5.14) and (5.17), the time evolution $u_{\theta_0}(k, \cdot, t)$ further decomposes in the sum

$$u_{\theta_0}(k, \cdot, t) = \sum_{j=1}^4 \psi_j(k, \cdot, t), \quad (5.18)$$

with

$$\psi_1(k, \cdot, t) = e^{-i\frac{t}{\varepsilon}k^2} \left[U_{\theta_0} \tilde{\psi}_-(k, \cdot) + C(k, t) (G^{k^2} - G(t)) \right], \quad (5.19)$$

$$\psi_2(k, \cdot, t) = -\int_0^t S^\varepsilon(t, s) \dot{C}(k, s) e^{-i\frac{s}{\varepsilon}k^2} (G^{k^2} - G(s)) ds, \quad (5.20)$$

$$\psi_3(k, \cdot, t) = -\int_0^t \dot{C}(k, s) e^{-i\frac{s}{\varepsilon}k^2} \left(S^\varepsilon(t, s) G(s) - \mu(t) e^{-\frac{i}{\varepsilon} \int_s^t E(\sigma) d\sigma} G(t) \right) ds, \quad (5.21)$$

$$\begin{aligned} \psi_4(k, \cdot, t) &= e^{-i\frac{t}{\varepsilon}k^2} \left[C(k, t) - \int_0^t \dot{C}(k, s) e^{-i\frac{s}{\varepsilon} \int_s^t (E(\sigma) - k^2) d\sigma} ds \right] \mu(t) G(t) \\ &\quad + e^{-i\frac{t}{\varepsilon}k^2} (1 - \mu(t)) C(k, t) G(t). \end{aligned} \quad (5.22)$$

The variable $A_{\theta_0}(t)$, associated with $u_{\theta_0}(k, \cdot, t)$, now writes as

$$A_{\theta_0}(t) = \sum_{j, j'=1}^4 \int \frac{dk}{2\pi h} g(k) \langle \chi \psi_j(k, \cdot, t), \psi_{j'}(k, \cdot, t) \rangle_{L^2(\mathbb{R})}. \quad (5.23)$$

In order to get adiabatic estimates for the contributions to (5.23), we need accurate asymptotic expansions for the quantities involved in these computations, including: $\tilde{\psi}_-(k, \cdot)$, and the integrals of $|C(k, t)|^2$. To this aim we introduce the following technical lemma.

Lemma 5.1 *In the assumptions (h1)-(h4), let $E \in \mathbb{R}$, $E^{\frac{1}{2}} \in \text{supp} g$ and $\lambda_t = \lim_{h \rightarrow 0} E(t)$; the solutions to (3.7) for $k^2 = E$ fulfill the conditions*

$$\left| \tilde{\psi}_-(E^{\frac{1}{2}}, c) \right|^2 = e^{-\frac{|\alpha_t|}{h}(c-a)} \mathcal{O}(1); \quad \left| \tilde{\psi}_-(-E^{\frac{1}{2}}, c) \right|^2 = e^{-\frac{|\alpha_t|}{h}(b-c)} \mathcal{O}(1), \quad (5.24)$$

for all $c \in (a, b)$. In particular, for $|E - \lambda_t| < C\varepsilon$ and $d(c, \{a, b\}) = c - a$, the first of (5.24) is explicitly

$$\left| \tilde{\psi}_-(E^{\frac{1}{2}}, c) \right|^2 = 2\lambda_t^{\frac{1}{2}} |\alpha_t| \frac{\Gamma_t}{|M(\lambda_t, \lambda_t)|^2} (1 + \mathcal{O}(|\theta_0|)) + o(\varepsilon), \quad (5.25)$$

$M(E_1, E_2)$, Γ_t and ε being defined according to (4.8), (5.2) and (2.18) respectively. The function $\tilde{\psi}_-(E^{\frac{1}{2}}, c)$ holomorphically extends to the complex neighbourhood $\mathbb{C} \cap \left\{ |z - E_{res}^h| \leq \frac{h}{d_0} \right\}$ where the representation

$$\tilde{\psi}_-(E^{\frac{1}{2}}, c) \tilde{\psi}_-^*((E^*)^{\frac{1}{2}}, c) = e^{-\frac{|\alpha|}{h}(c-a)} \mathcal{F}^{h, \theta_0}(E) \quad (5.26)$$

holds, being $\mathcal{F}^{h, \theta_0}(\cdot)$ an holomorphic family uniformly bounded w.r.t. h and θ_0 .

Proof. In (3.10) the explicit form of $\tilde{\psi}_-(k, \cdot)$, $k > 0$, is given. For energies k^2 placed below the barrier level V_0 , the decreasing behaviour of the terms $e^{-i\frac{\Lambda_z}{h}l}$, $e^{-i\frac{\Lambda_z}{h}(c-b)}$ w.r.t. h allow to write

$$\tilde{\psi}_-(k, c) = e^{-i\frac{\Lambda_z}{h}(c-a)} \mathcal{P}^h(k^2, \theta_0) \quad (5.27)$$

with Λ_z defined as in (3.11), $\mathcal{P}^h(z, \theta_0)$ an holomorphic map w.r.t. both variables, provided that $|z - E_{res}^h| < \frac{h}{d_0}$ and $|\theta_0|, h$ are small enough. The first part of (5.24) follow by using (5.27) with: $k^2 = E = \lambda_t + \mathcal{O}(h)$. The second part of (5.24) can be carried out by a similar direct computation. For $h \rightarrow 0$, the asymptotic behaviour of $\tilde{\psi}_-(k, c)$ is determined by the factor $e^{-i\frac{\Lambda_z}{h}(c-a)}$. In the complex neighbourhood $|z - E_{res}^h| < \frac{h}{d_0}$, where $E_{res}^h = V_0 - \frac{\alpha^2}{4} + \mathcal{O}\left(e^{-\frac{|\alpha_t|}{h}d(c, \{a, b\})}\right)$, the function Λ_z is analytic and the relation: $e^{-i\frac{\Lambda_z}{h}(c-a)} = e^{-\frac{|\alpha|}{2h}(c-a)} \mathcal{R}^h(z)$ holds being $\mathcal{R}^h(\cdot)$ an analytic family uniformly bounded w.r.t. h . A similar identity holds for $\tilde{\psi}_-(k, c)$, once (5.27) is taken into account. The representation (5.26) is a direct consequence of this relation.

Next we use the notation $\tilde{\psi}_{-, \theta_0}(k, \cdot)$ and $G_{\theta_0}^E(\cdot, c)$ to point out the dependence of scattering states and Green's functions on the interface conditions of the Hamiltonian. When $\theta_0 = 0$, $H_{0,0}^h(0)$ is a selfadjoint operator with purely absolutely continuous spectrum. The Stone's formula yields in this case

$$\begin{aligned} &\int_0^{+\infty} \frac{dk}{2\pi h} f(k^2) \left[\left| \tilde{\psi}_{-,0}(k, c) \right|^2 + \left| \tilde{\psi}_{-,0}(-k, c) \right|^2 \right] \\ &= \frac{1}{2\pi i} \lim_{\delta \rightarrow 0} \int_0^{+\infty} dE f(E) \left\langle \delta_c, \left[(H_{0,0}^h(0) - E + i\delta)^{-1} - (H_{0,0}^h(0) - E - i\delta)^{-1} \right] \delta_c \right\rangle_{H^{-1}, H^1} \\ &= \frac{1}{\pi} \lim_{\delta \rightarrow 0} \int dE f(E) \text{Im} G_0^{E-i\delta}(c, c) = \frac{1}{\pi} \lim_{\delta \rightarrow 0} \int dk 2|k| f(k^2) \text{Im} G_0^{k^2-i\delta}(c, c). \end{aligned}$$

for continuous f . This leads to

$$\left| \tilde{\psi}_{-,0}(k, c) \right|^2 + \left| \tilde{\psi}_{-,0}(-k, c) \right|^2 = 4h |k| \operatorname{Im} G_0^{k^2 - i0}(c, c). \quad (5.28)$$

For $|E - \lambda_t| < C\varepsilon$, (5.24) implies: $\left| \tilde{\psi}_{-,0}(-E^{\frac{1}{2}}, c) \right|^2 = \mathcal{O}(e^{-\frac{|\alpha_t|}{h}(b-c)})$, and due to the assumption $d(c, \{a, b\}) = c - a$, we get

$$\left| \tilde{\psi}_{-,0}(E^{\frac{1}{2}}, c) \right|^2 = 4h E^{\frac{1}{2}} \operatorname{Im} G_0^{E - i0}(c, c) + o(\varepsilon). \quad (5.29)$$

The relation (5.25), for $\theta_0 = 0$, follows by using (4.7) and (4.10) to express $\operatorname{Im} G_0^{E - i0}(c, c)$ as a function of $M(E, E(t))$, and expanding for $E = \lambda_t - i\Gamma_t + o(\varepsilon)$. The general case is recovered by noticing that (5.27), and the correspondent expression for $\tilde{\psi}_{-, \theta_0}(-k, \cdot)$, imply

$$\left| \tilde{\psi}_{-, \theta_0}(k, c) \right|^2 + \left| \tilde{\psi}_{-, \theta_0}(-k, c) \right|^2 = \left(\left| \tilde{\psi}_{-,0}(k, c) \right|^2 + \left| \tilde{\psi}_{-,0}(-k, c) \right|^2 \right) (1 + \mathcal{O}(|\theta_0|)). \quad (5.30)$$

(Actually (5.30) could be also recovered, with the less efficient bound $\mathcal{O}(h^{-1}|\theta_0|)$, from Propositions 4.5 in [11]). Thus, for $|\theta_0| \ll 1$ (we refer to the assumption (h1)), (5.28) writes as

$$\left| \tilde{\psi}_{-, \theta_0}(k, c) \right|^2 + \left| \tilde{\psi}_{-, \theta_0}(-k, c) \right|^2 = 4h |k| \operatorname{Im} G_0^{k^2 - i0}(c, c) (1 + \mathcal{O}(|\theta_0|)). \quad (5.31)$$

Proceeding as before, we obtain (5.25). ■

Next computations involve the use of small- h expansions of the coefficients $C(k, t)$ and $\dot{C}(k, t)$. Using the definition (5.13) and the relation (4.7), leads

$$C(k, t) = -\frac{h\alpha_t \tilde{\psi}_{-}(k, c) M(k^2, E(t))}{k^2 - E(t)}. \quad (5.32)$$

The derivative $\dot{C}(k, s)$ is explicitly given by

$$\dot{C}(k, t) = \frac{h \dot{\alpha}_t \tilde{\psi}_{-}(k, c)}{(1 + h\alpha_t G^{k^2}(c))^2} (2h\alpha_t G^{k^2}(c) - 1) = \frac{h \dot{\alpha}_t \tilde{\psi}_{-}(k, c) M(k^2, E(t))}{(k^2 - E(t))^2} \mathcal{O}(1) \quad (5.33)$$

for all $k \in \operatorname{supp} g$, and $t \in [0, T]$. The relations (5.24) and (5.25), allows to identify $|C(k, t)|^2$ with a Lorentzian function on $\operatorname{supp} g$, with scale parameter given by Γ_t . In particular, for: $d(c, \{a, b\}) = c - a$, it holds

$$\int \frac{dk}{2\pi h} g(k) |C(k, t)|^2 = \frac{h |\alpha_t|^3}{2} g\left(\lambda_t^{\frac{1}{2}}\right) (1 + \mathcal{O}(|\theta_0|)) + o(\varepsilon), \quad (5.34)$$

while, for $d(c, \{a, b\}) = b - c$, we have

$$\int \frac{dk}{2\pi h} g(k) |C(k, t)|^2 = \mathcal{O}\left(e^{-\frac{\beta}{h}}\right), \quad (5.35)$$

with positive $\beta = \frac{|\alpha_t|}{h}(c - a - (b - c))$. Both the above expansions follow by using the dilation $y = \frac{k^2 - E(t)}{\Gamma_t}$ and taking the limit of the resulting integral as $h \rightarrow 0$. With a similar computation we also have

$$\int dk g(k) \frac{1}{|k^2 - E(t)|} = \mathcal{O}\left(\frac{1}{h}\right). \quad (5.36)$$

Lemma 5.2 *In the assumptions (h1)-(h4), the estimates:*

$$\left| \int \frac{dk}{2\pi h} g(k) \langle \chi \psi_j(k, \cdot, t), \psi_j(k, \cdot, t) \rangle_{L^2(\mathbb{R})} \right| = \tilde{\mathcal{O}}(\varepsilon^{\frac{1}{J+\delta}}) \quad (5.37)$$

hold with: $j = 1, 2, 3$.

Proof. For $j = 1$, this product develops in the sum

$$\begin{aligned} & \int \frac{dk}{2\pi h} g(k) \left\langle \chi U_{\theta_0} \tilde{\psi}_-(k, \cdot), U_{\theta_0} \tilde{\psi}_-(k, \cdot) \right\rangle_{L^2(\mathbb{R})} \\ & + \int \frac{dk}{2\pi h} g(k) \left\langle \chi C(k, t) \left(G^{k^2} - G(t) \right), C(k, t) \left(G^{k^2} - G(t) \right) \right\rangle_{L^2(\mathbb{R})} + \\ & 2 \operatorname{Re} \int \frac{dk}{2\pi h} g(k) \left\langle \chi U_{\theta_0} \tilde{\psi}_-(k, \cdot), C(k, t) \left(G^{k^2} - G(t) \right) \right\rangle_{L^2(\mathbb{R})} . \end{aligned}$$

Using the exponential decreasing behaviour of $\tilde{\psi}_-(k, \cdot)$ on the $\operatorname{supp} \chi$ (see Lemma 5.1), the first contribution is estimated by

$$\left| \int \frac{dk}{2\pi h} g(k) \left\langle \chi U_{\theta_0} \tilde{\psi}_-(k, \cdot), U_{\theta_0} \tilde{\psi}_-(k, \cdot) \right\rangle_{L^2(\mathbb{R})} \right| = \tilde{\mathcal{O}} \left(e^{-2 \frac{|\alpha_0|}{h} (c-a)} \right) . \quad (5.38)$$

For the second term, the definition of $C(k, t)$ (see (5.32)), the equivalence (5.24) and the first inequality in (3.20) lead to

$$\begin{aligned} & \left| \int \frac{dk}{2\pi h} g(k) |C(k, t)|^2 \left\langle \chi \left(G^{k^2} - G(t) \right), \left(G^{k^2} - G(t) \right) \right\rangle_{L^2(\mathbb{R})} \right| \\ & \leq C \int \frac{dk}{2\pi h} |g(k)| |C(k, t)|^2 \left\| G^{k^2} - G(t) \right\|_{L^2(\mathbb{R})} = \tilde{\mathcal{O}} \left(e^{-\frac{|\alpha_0|}{h} (c-a)} \right) . \end{aligned} \quad (5.39)$$

The last contribution is a crossing term; it is estimated in terms of the previous ones by using the Hölder inequality in $L^2(\mathbb{R}^2)$

$$\left| \int \frac{dk}{2\pi h} g(k) \left\langle \chi U_{\theta_0} \tilde{\psi}_-(k, \cdot), C(k, t) \left(G^{k^2} - G(t) \right) \right\rangle_{L^2(\mathbb{R})} \right| \leq \tilde{\mathcal{O}}(\varepsilon) . \quad (5.40)$$

For $j = 2$, the integral is

$$\int \frac{dk}{2\pi h} g(k) \langle \chi \psi_2(k, \cdot, t), \psi_2'(k, \cdot, t) \rangle_{L^2(\mathbb{R})} = \int \frac{dk}{2\pi h} \int_0^t ds_1 \int_0^t ds_2 f(k, s_1, s_2) , \quad (5.41)$$

with

$$\begin{aligned} f(k, s_1, s_2) &= g(k) \dot{C}(k, s_1) \dot{C}^*(k, s_2) e^{-i \frac{(s_1 - s_2)}{\varepsilon} k^2} \times \\ & \times \left\langle \chi S^\varepsilon(t, s_1) \left(G^{k^2} - G(s_1) \right), S^\varepsilon(t, s_2) \left(G^{k^2} - G(s_2) \right) \right\rangle_{L^2(\mathbb{R})} . \end{aligned} \quad (5.42)$$

Since $f(k, s_1, s_2) = f^*(k, s_2, s_1)$, this integral writes as

$$\int \frac{dk}{2\pi h} g(k) \langle \chi \psi_2(k, \cdot, t), \psi_2'(k, \cdot, t) \rangle_{L^2(\mathbb{R})} = 2 \operatorname{Re} \int \frac{dk}{2\pi h} \int_0^t ds_1 \int_0^{s_1} ds_2 f(k, s_1, s_2) . \quad (5.43)$$

The first inequality of (3.20), leads: $\left\| S^\varepsilon(t, s) \left(G^{k^2} - G(s) \right) \right\|_{L^2(\mathbb{R})} \leq |k^2 - E(s)| \tilde{\mathcal{O}}(\varepsilon^0)$. Then, according to the definitions (5.32)-(5.33), we find

$$|f(k, s_1, s_2)| \leq \tilde{\mathcal{O}}(\varepsilon^0) |g(k)| |C(k, s_1) C^*(k, s_2)| ,$$

and

$$\left| \int \frac{dk}{2\pi h} \int_0^t ds_1 \int_0^{s_1} ds_2 f(k, s_1, s_2) \right| \leq \tilde{\mathcal{O}}(\varepsilon^0) \int_0^t ds_1 \int_0^{s_1} ds_2 \int dk \left| \frac{g(k)}{2\pi h} \right| |C(k, s_1) C^*(k, s_2)| . \quad (5.44)$$

The integral over k admits two independent estimates:

1) Use the Cauchy-Schwarz inequality to write

$$\begin{aligned} & \int dk \left| \frac{g(k)}{2\pi h} \right| |C(k, s_1) C^*(k, s_2)| \\ & \leq \left(\int \frac{dk}{2\pi h} g(k) |C(k, s_1)|^2 \right)^{\frac{1}{2}} \left(\int \frac{dk}{2\pi h} g(k) |C(k, s_2)|^2 \right)^{\frac{1}{2}} = \mathcal{O}(1). \end{aligned} \quad (5.45)$$

2) Use (5.32) and the relation

$$\frac{|E(s_1) - E(s_2)|}{|k^2 - E(s_1)| |k^2 - E(s_2)|} = \left| \frac{1}{k^2 - E(s_1)} - \frac{1}{k^2 - E(s_2)} \right|$$

to write

$$\int dk \left| \frac{g(k)}{2\pi h} \right| |C(k, s_1) C^*(k, s_2)| \leq \frac{\tilde{\mathcal{O}}(\varepsilon^0)}{|E(s_1) - E(s_2)|} \left(\int_{\text{supp } g} dk \frac{|\tilde{\psi}_-(k, c)|^2}{|k^2 - E(s_1)|} + \int_{\text{supp } g} dk \frac{|\tilde{\psi}_-(k, c)|^2}{|k^2 - E(s_2)|} \right).$$

From (5.36) and (5.24), it follows

$$\int dk \left| \frac{g(k)}{2\pi h} \right| |C(k, s_1) C^*(k, s_2)| \leq \frac{\tilde{\mathcal{O}} \left(e^{-\frac{|\alpha_t|}{n} (c-a)} \right)}{|E(s_1) - E(s_2)|}. \quad (5.46)$$

Interpolating between (5.45) and (5.46) yields

$$\int dk \left| \frac{g(k)}{2\pi h} \right| |C(k, s_1) C^*(k, s_2)| \leq \frac{\tilde{\mathcal{O}} \left(e^{-\frac{|\alpha_t|}{n} (c-a)} \right)}{|E(s_1) - E(s_2)|^{\frac{1}{n}}} \leq \frac{\tilde{\mathcal{O}} \left(e^{-\frac{|\alpha_t|}{n} (c-a)} \right)}{|\alpha(s_1) - \alpha(s_2)|^{\frac{1}{n}}},$$

where we use the lower bound: $|E(s_1) - E(s_2)| \geq c_0 |\alpha(s_1) - \alpha(s_2)|$ following from (2.20)-(2.21) and the analyticity of the map $\alpha \rightarrow E_{res}^h$ (see Proposition 4.1). Due to the assumption (h2), $|\alpha(s_1) - \alpha(s_2)|^{-\frac{1}{n}}$ is integrable on the triangle $\{s_1 \in [0, T], s_2 \leq s_1\}$ provided that $n \geq J + \delta$. This leads to

$$\left| \int \frac{dk}{2\pi h} g(k) \langle \chi \psi_2(k, \cdot, t), \psi_{2'}(k, \cdot, t) \rangle_{L^2(\mathbb{R})} \right| = \tilde{\mathcal{O}} \left(e^{-\frac{|\alpha_t|}{n} (c-a)} \right)$$

For $j = 3$, the integral is

$$\begin{aligned} & \int \frac{dk}{2\pi h} g(k) \langle \chi \psi_3(k, \cdot, t), \psi_{3'}(k, \cdot, t) \rangle_{L^2(\mathbb{R})} \\ & = \int_0^t ds_1 \int_0^t ds_2 \int \frac{dk}{2\pi h} g(k) \dot{C}(k, s_1) \dot{C}^*(k, s_2) \langle \chi \varphi(t, s_1), \varphi(t, s_2) \rangle_{L^2(\mathbb{R})}, \end{aligned}$$

where

$$\varphi(t, s) = S^\varepsilon(t, s) G(s) - \mu(t) e^{-\frac{i}{\varepsilon} \int_s^t E(\sigma) d\sigma} G(t).$$

Using (5.7), (5.8) and (5.9), it follows: $|\varphi(t, s)|_{L^2(\mathbb{R})} \leq \tilde{\mathcal{O}}(\varepsilon)$ uniformly w.r.t. t and s ; then, proceeding as above, we get

$$\left| \int \frac{dk}{2\pi h} g(k) \langle \chi \psi_3(k, \cdot, t), \psi_{3'}(k, \cdot, t) \rangle_{L^2(\mathbb{R})} \right| \leq \tilde{\mathcal{O}}(\varepsilon^2) \int_0^t ds_1 \int_0^t ds_2 \int dk \left| \frac{g(k)}{2\pi h} \right| \frac{|C(k, s_1) C^*(k, s_2)|}{|k^2 - E(s_1)| |k^2 - E(s_2)|}.$$

Since $|k^2 - E(s)|^{-1} \leq \frac{1}{\varepsilon}$ on $\text{supp } g$, a similar inequality to the one considered in (5.44) follows.

We obtain

$$\left| \int \frac{dk}{2\pi h} g(k) \langle \chi \psi_3(k, \cdot, t), \psi_{3'}(k, \cdot, t) \rangle_{L^2(\mathbb{R})} \right| \leq \tilde{\mathcal{O}} \left(e^{-\frac{|\alpha_t|}{n} (c-a)} \right).$$

■

5.2 The reduced equation.

We consider the term

$$\int \frac{dk}{2\pi h} g(k) \langle \chi \psi_4(k, \cdot, t), \psi_{4'}(k, \cdot, t) \rangle . \quad (5.47)$$

Setting: $\psi_4(k, \cdot, t) = \varphi_1(k, \cdot, t) + \varphi_2(k, \cdot, t)$,

$$\varphi_1(k, \cdot, t) = e^{-i\frac{t}{\varepsilon}k^2} \left[C(k, t) - \int_0^t \dot{C}(k, s) e^{-\frac{i}{\varepsilon} \int_s^t (E(\sigma) - k^2) d\sigma} ds \right] \mu(t) G(t) , \quad (5.48)$$

$$\varphi_2(k, \cdot, t) = e^{-i\frac{t}{\varepsilon}k^2} (1 - \mu(t)) C(k, t) G(t) , \quad (5.49)$$

and introducing the variables

$$a(t) = \int \frac{dk}{2\pi h} g(k) \langle \chi \varphi_1(k, \cdot, t), \varphi_1(k, \cdot, t) \rangle , \quad (5.50)$$

$$\mathcal{J}_1(t) = \int \frac{dk}{2\pi h} g(k) \langle \chi \varphi_2(k, \cdot, t), \varphi_2(k, \cdot, t) \rangle , \quad (5.51)$$

$$\mathcal{J}_2(t) = 2 \operatorname{Re} \int \frac{dk}{2\pi h} g(k) \langle \chi \varphi_1(k, \cdot, t), \varphi_2(k, \cdot, t) \rangle , \quad (5.52)$$

it becomes

$$\int \frac{dk}{2\pi h} g(k) \langle \chi \psi_4(k, \cdot, t), \psi_{4'}(k, \cdot, t) \rangle = a(t) + \mathcal{J}_1(t) + \mathcal{J}_2(t) . \quad (5.53)$$

In what follows the asymptotic analysis of these contribution as $h \rightarrow 0$ is developed. Let start with $a(t)$: it can be rephrased as

$$a(t) = \int \frac{dk}{2\pi h} g(k) |\beta(k, t)|^2 |\mu(t)|^2 \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} , \quad (5.54)$$

$$\beta(k, t) = C(k, t) - \int_0^t \dot{C}(k, s) e^{-\frac{i}{\varepsilon} \int_s^t (E(\sigma) - k^2) d\sigma} ds . \quad (5.55)$$

According to the definitions of μ (see (5.9)) and β , we have

$$\partial_t \mu(t) = -\frac{\langle G(t), \partial_t G(t) \rangle}{\|G(t)\|_{L^2(\mathbb{R})}^2} \mu(t) + \tilde{\mathcal{O}}(\varepsilon) , \quad (5.56)$$

$$\partial_t \beta(k, t) = -\frac{i}{\varepsilon} (E(t) - k^2) (\beta(k, t) - C(k, t)) . \quad (5.57)$$

Out of exponentially small terms, this leads to the differential relation

$$\begin{aligned} \partial_t a(t) &= \left[-2 \operatorname{Re} \frac{\langle G(t), \partial_t G(t) \rangle}{\|G(t)\|_{L^2(\mathbb{R})}^2} + \partial_t \ln \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} \right] a(t) \\ &\quad - 2 \operatorname{Re} \frac{i}{\varepsilon} \int \frac{dk}{2\pi h} g(k) (E(t) - k^2) |\beta(k, t)|^2 |\mu(t)|^2 \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} \\ &\quad + 2 \operatorname{Re} \frac{i}{\varepsilon} \int \frac{dk}{2\pi h} g(k) (E(t) - k^2) |\mu(t)|^2 \bar{\beta}(k, t) C(k, t) \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} . \end{aligned} \quad (5.58)$$

Using $E(t) - k^2 = (E_R(t) - k^2) - i\Gamma_t$, it follows

$$\partial_t a(t) = \left[-2 \operatorname{Re} \frac{\langle G(t), \partial_t G(t) \rangle}{\|G(t)\|_{L^2(\mathbb{R})}^2} + \partial_t \ln \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} - 2 \frac{\Gamma_t}{\varepsilon} \right] a(t) + \mathcal{S}^h(t) , \quad (5.59)$$

$$\mathcal{S}^h(t) = 2 \operatorname{Re} \frac{i}{\varepsilon} \int \frac{dk}{2\pi h} g(k) (E(t) - k^2) |\mu(t)|^2 \bar{\beta}(k, t) C(k, t) \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} . \quad (5.60)$$

The derivative at the l.h.s. is explicitly given by

$$\partial_t \ln \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} = 2 \operatorname{Re} \frac{\langle \chi G(t), \partial_t G(t) \rangle}{\|G(t)\|_{L^2(\mathbb{R})}^2}, \quad (5.61)$$

so we get

$$\partial_t a(t) = \left[\frac{2}{\|G(t)\|_2^2} \operatorname{Re} \langle (\chi - 1) G(t), \partial_t G(t) \rangle - 2 \frac{\Gamma_t}{\varepsilon} \right] a(t) + \mathcal{S}^h(t). \quad (5.62)$$

We next discuss the small- h behaviour of the source term. This can be further developed as $\mathcal{S}^h = \mathcal{S}_1^h + \mathcal{S}_2^h$, with

$$\mathcal{S}_1^h(t) = \mathcal{W}(t) \frac{\Gamma_t}{\varepsilon} \int \frac{dk}{2\pi h} g(k) |C(k, t)|^2, \quad (5.63)$$

$$\mathcal{S}_2^h(t) = \mathcal{W}(t) \operatorname{Re} \int \frac{dk}{2\pi h} g(k) \frac{i}{\varepsilon} (k^2 - E(t)) C(k, t) \int_0^t \dot{C}^*(k, s) e^{-\frac{i}{\varepsilon} \int_s^t (k^2 - E^*(\sigma)) d\sigma} ds, \quad (5.64)$$

and $\mathcal{W}(t) = 2 |\mu(t)|^2 \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})}$. Since

$$|\mu(t)| = \frac{\|G(0)\|_{L^2(\mathbb{R})}}{\|G(t)\|_{L^2(\mathbb{R})}} + \tilde{\mathcal{O}}(\varepsilon), \quad (5.65)$$

the exponentially decreasing character of the Green's functions outside $\operatorname{supp} \chi$ (see Lemma 3.2) and the relation (4.11) lead to

$$\begin{aligned} \mathcal{W}(t) &= 2 \frac{\langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})}}{\langle G(t), G(t) \rangle_{L^2(\mathbb{R})}} \|G(0)\|_{L^2(\mathbb{R})}^2 + \tilde{\mathcal{O}}(\varepsilon) = 2 \|G(0)\|_{L^2(\mathbb{R})}^2 + \tilde{\mathcal{O}}(\varepsilon) \\ &= \frac{2}{h \alpha_0 M(E(0), E(0))} + \tilde{\mathcal{O}}(\varepsilon) = \frac{4}{h |\alpha_0|^3} + \tilde{\mathcal{O}}(\varepsilon). \end{aligned} \quad (5.66)$$

Thus, $\frac{h|\alpha_0|^3}{4} \mathcal{S}_1^h(t)$ expands as (5.34) or (5.35), depending on the value of $d(c, \{a, b\})$. For the first contribution to \mathcal{S}^h we get

$$\mathcal{S}_1^h(t) = 2 \left| \frac{\alpha_t}{\alpha_0} \right|^3 \frac{\Gamma_t}{\varepsilon} g\left(\lambda_t^{\frac{1}{2}}\right) (1 + \mathcal{O}(|\theta_0|)) + o(\varepsilon), \quad \text{for } d(c, \{a, b\}) = c - a \quad (5.67)$$

$$\mathcal{S}_1^h(t) = \mathcal{O}\left(e^{-\frac{2\beta}{h}}\right), \quad \text{for } d(c, \{a, b\}) = b - c \quad (5.68)$$

with $\beta = \frac{|\alpha_t|}{h} (c - a - (b - c))$. After changing the variable: $E = k^2$, the second contribution writes as

$$\mathcal{S}_2^h(t) = \mathcal{W}(t) \operatorname{Re} \int F(E, t) dE, \quad (5.69)$$

$$F(E, t) = \frac{1}{2E^{\frac{1}{2}}} g(E^{\frac{1}{2}}) \frac{i}{\varepsilon} (E - E(t)) C(E^{\frac{1}{2}}, t) \int_0^t \dot{C}^*((E^*)^{\frac{1}{2}}, s) e^{-\frac{i}{\varepsilon} \int_s^t (E - E^*(\sigma)) d\sigma} ds. \quad (5.70)$$

According to the assumption (h3), $F(\cdot, t)$ extends to an holomorphic function of $E \in \left\{ z \in \mathbb{C}, |z - \lambda_0| < \frac{h}{d_0} \right\}$, while for $E \in \operatorname{supp} g(E^{\frac{1}{2}}) \setminus \left\{ |E - \lambda_0| < \frac{h}{d_0} \right\}$, the definitions (5.32), (5.33) and the exponential bounds (5.24) imply: $|F(E, t)| = \tilde{\mathcal{O}}(\varepsilon)$. In particular, the term: $(E - E(t)) C(E, t)$ is analytic in a complex neighbourhood of $\lambda_0 = \lim_{h \rightarrow 0} E(0)$, while $\dot{C}^*((E^*)^{\frac{1}{2}}, s)$ is meromorphic with a double pole at $E = E^*(s)$, placed in the upper half plane. Our strategy is to use a complex integration path formed by the semi-circumference $\mathcal{C}_{\frac{h}{d_0}}(\lambda_0)$ of center λ_0 , radius $\frac{h}{d_0}$ in the lower half-plane:

$\text{Im } \mathcal{C}_{\frac{h}{d_0}}(\lambda_0) \leq 0$. Let us consider an holomorphic extension of $F(\cdot, t)$ to the half-disk whose boundary is determined by: $\left\{ E \in \mathbb{R}, |E - \lambda_0| < \frac{h}{d_0} \right\} \cup \mathcal{C}_{\frac{h}{d_0}}(\lambda_0)$. Using (5.32), (5.33), (5.26) and the function $\vartheta(\cdot)$

$$\vartheta(z) = |\text{Im } z|, \quad (5.71)$$

the restriction of $F(\cdot, t)$ to $\mathcal{C}_{\frac{h}{d_0}}(\lambda_0)$ is bounded by

$$F(\cdot, t)|_{\mathcal{C}_{\frac{h}{d_0}}(\lambda_0)} \leq \frac{C}{\varepsilon} e^{-\frac{|\alpha_t|}{h}(c-a)} \int_0^t e^{-\frac{\vartheta(E)}{\varepsilon}(t-s)} ds \leq C \int_0^t e^{-\frac{\vartheta(E)}{\varepsilon}(t-s)} ds. \quad (5.72)$$

for a suitable positive C . According to (5.72), the following estimates hold

1. $F(\cdot, t)|_{\mathcal{C}_{\frac{h}{d_0}}(\lambda_0)} \leq C \frac{\varepsilon}{\vartheta(E)} \left[1 - e^{-\frac{\vartheta(E)}{\varepsilon}t} \right] \leq C \frac{\varepsilon}{\vartheta(E)},$
2. $F(\cdot, t)|_{\mathcal{C}_{\frac{h}{d_0}}(\lambda_0)} \leq Ct,$

and by interpolation we obtain

$$F(\cdot, t)|_{\mathcal{C}_{\frac{h}{d_0}}(\lambda_0)} \leq C \frac{\varepsilon^{\frac{1}{2}}}{\vartheta^{\frac{1}{2}}(E)}. \quad (5.73)$$

By computing the residue, $\mathcal{S}_2^h(t)$ writes as

$$\mathcal{S}_2^h(t) = -\mathcal{W}(t) \int_{\mathcal{C}_{\frac{h}{d_0}}(\lambda_0)} F(E, t) dE + \tilde{\mathcal{O}}(\varepsilon). \quad (5.74)$$

Denoting $z \in \mathcal{C}_{\frac{h}{d_0}}(\lambda_0)$ as: $z = \lambda_0 + \frac{h}{d} e^{i\varphi}$, $\omega \in (-\pi, 0)$, the previous inequality implies

$$\sup_t |\mathcal{S}_2^h(t)| \leq C \varepsilon^{\frac{1}{2}} \int_{\mathcal{C}_{\frac{h}{d_0}}(\lambda_0)} \frac{|dE|}{\vartheta^{\frac{1}{2}}(E)} = C \varepsilon^{\frac{1}{2}} \frac{h}{d} \int_{-\pi}^0 \frac{d\omega}{\sin^{\frac{1}{2}} \omega} = \mathcal{O}(h \varepsilon^{\frac{1}{2}}). \quad (5.75)$$

The estimates (5.67)-(5.68) and (5.75), allow to use $\mathcal{S}^h = \mathcal{S}_1^0 + \mathcal{O}(|\theta_0|) + \mathcal{O}(h \varepsilon^{\frac{1}{2}})$, with

$$\mathcal{S}_1^0 = 2 \left| \frac{\alpha_t}{\alpha_0} \right|^3 \frac{\Gamma_t}{\varepsilon} g\left(\lambda_t^{\frac{1}{2}}\right), \quad \text{for } d(c, \{a, b\}) = c - a, \quad (5.76)$$

$$\mathcal{S}_1^0 = \mathcal{O}\left(e^{-\frac{\beta}{h}}\right), \quad \text{for } d(c, \{a, b\}) = b - c, \quad (5.77)$$

with: $\beta = \frac{|\alpha_t|}{h}(c-a-(b-c))$. Owing to the estimates in Lemma 3.2, the term $\frac{2}{|G(t)|_2^2} \text{Re} \langle (\chi - 1) G(t), \partial_t G(t) \rangle$, is bounded by

$$\frac{1}{|G(t)|_2^2} \left| \langle (\chi - 1) G(t), \partial_t G(t) \rangle_{L^2(\mathbb{R})} \right| \leq C |(\chi - 1) G(t)|_{L^2(\mathbb{R})} |\partial_t G(t)|_{L^2(\mathbb{R})} = \tilde{\mathcal{O}} \left(\inf_{\supp (1-\chi)} e^{-\frac{|\alpha_0|}{2h} |\cdot - c|} \right). \quad (5.78)$$

When the interaction point 'c' is on the left side of the barrier's support and the condition $d(c, \{a, b\}) = c - a$ is fulfilled, the limit condition (5.76) and the estimates (5.75), (5.78) allow to write (5.62) as follows

$$\partial_t a(t) = \left(-2 \frac{\Gamma_t}{\varepsilon} \right) \left(a(t) - \left| \frac{\alpha_t}{\alpha_0} \right|^3 g\left(\lambda_t^{\frac{1}{2}}\right) \right) + \mathcal{O}(|\theta_0|) + \tilde{\mathcal{O}}\left(e^{-\frac{\tau_\chi}{h}}\right), \quad (5.79)$$

where $\tau_\chi > 0$ is defined according to the remainders in (5.67), (5.75) and (5.78). The initial datum for this equation is deduced by evaluating (5.54) at $t = 0$. With the above expansions, we obtain

$$a(0) = g\left(\lambda_0^{\frac{1}{2}}\right) (1 + \mathcal{O}(|\theta_0|)) + o(\varepsilon), \quad \text{for } d(c, \{a, b\}) = c - a. \quad (5.80)$$

When $d(c, \{a, b\}) = c - a$, the solution $a(t)$ is

$$\begin{cases} a(t) = a(0)e^{-2 \int_0^t \frac{\Gamma_s}{\varepsilon} ds} + \int_0^t e^{-2 \int_s^t \frac{\Gamma_\sigma}{\varepsilon} d\sigma} \mathcal{S}_1^0(s) ds + \mathcal{O}(|\theta_0|) + \tilde{\mathcal{O}}\left(e^{-\frac{\tau_X}{h}}\right), \\ a(0) = g\left(\lambda_0^{\frac{1}{2}}\right); \quad \mathcal{S}_1^0(t) = 2 \left| \frac{\alpha_t}{\alpha_0} \right|^3 \frac{\Gamma_t}{\varepsilon} g\left(\lambda_t^{\frac{1}{2}}\right). \end{cases} \quad (5.81)$$

In the other case, when $d(c, \{a, b\}) = b - c$, the initial value of $a(t)$ is: $a(0) = \mathcal{O}\left(e^{-\frac{\beta}{h}}\right)$ which coincides with the size of the source term given in (5.77). This leads to

$$a(t) = \mathcal{O}\left(e^{-\frac{\beta}{h}}\right), \quad \text{for } d(c, \{a, b\}) = b - c. \quad (5.82)$$

with $\beta = \frac{|\alpha_t|}{h}(c - a - (b - c))$.

To complete the proof of the second point of Theorem 2.1, we need the following Lemma.

Lemma 5.3 *In the assumptions (h1)-(h4), the relations*

$$\int \frac{dk}{2\pi h} g(k) \langle \chi \psi_j(k, \cdot, t), \psi_{j'}(k, \cdot, t) \rangle_{L^2(\mathbb{R})} = \tilde{\mathcal{O}}(\varepsilon^{\frac{1}{2}}), \quad (5.83)$$

hold with: $j, j' = 1, 2, 3, 4$ and $j \neq j'$.

Proof. Let consider the contributions $J_{i=1,2}(t)$ to (5.47). The first term explicitly writes as

$$\mathcal{J}_1(t) = |1 - \mu(t)|^2 \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} \int \frac{dk}{2\pi h} g(k) |C(k, t)|^2. \quad (5.84)$$

Then, the estimate (3.17), and the relations (5.34), (5.65) yield: $\mathcal{J}_1(t) = \tilde{\mathcal{O}}(\varepsilon^0)$, holding for any choice of χ, g fulfilling the assumptions. For the second terms, let take $\tilde{g}, \tilde{\chi}$ a couple of positive functions fulfilling (h3), and such that: $|g| < \tilde{g}, |\chi| < \tilde{\chi}$. With this conditions, a straightforward application of the Cauchy-Schwartz inequality gives

$$\begin{aligned} \frac{|\mathcal{J}_2(t)|}{2} &\leq \left| \int \frac{dk}{2\pi h} g(k) \langle \chi \varphi_1(k, \cdot, t), \varphi_2(k, \cdot, t) \rangle \right| \leq \int \int \frac{dk dx}{2\pi h} \tilde{g}(k) \tilde{\chi}(x) |\varphi_1(k, x, t), \varphi_2(k, x, t)| \\ &\leq \left(\int \frac{dk}{2\pi h} \tilde{g}(k) \langle \tilde{\chi} \varphi_1(k, \cdot, t), \varphi_1(k, \cdot, t) \rangle \right)^{\frac{1}{2}} \left(\int \frac{dk}{2\pi h} \tilde{g}(k) \langle \tilde{\chi} \varphi_2(k, \cdot, t), \varphi_2(k, \cdot, t) \rangle \right)^{\frac{1}{2}} \\ &= \tilde{a}^{\frac{1}{2}}(t) \tilde{\mathcal{J}}_1^{\frac{1}{2}}(t), \end{aligned}$$

with \tilde{a} and $\tilde{\mathcal{J}}_1$ denoting the principal contribution and the first remainder arising from the auxiliary data $\tilde{g}, \tilde{\chi}$. Since $\tilde{a}(t) = \mathcal{O}(1)$ (as it follows from (5.81)) and $\tilde{\mathcal{J}}_1(t) = \tilde{\mathcal{O}}(\varepsilon^0)$, we obtain: $\mathcal{J}_2(t) = \tilde{\mathcal{O}}(\varepsilon^0)$. This leads to

$$\int \frac{dk}{2\pi h} g(k) \langle \chi \psi_4(k, \cdot, t), \psi_4(k, \cdot, t) \rangle_{L^2(\mathbb{R})} = \tilde{\mathcal{O}}(\varepsilon^0) \quad (5.85)$$

while the results of Lemma 5.2, gives

$$\int \frac{dk}{2\pi h} g(k) \langle \chi \psi_j(k, \cdot, t), \psi_j(k, \cdot, t) \rangle_{L^2(\mathbb{R})} = \tilde{\mathcal{O}}(\varepsilon), \quad \text{with } j = 1, 2, 3. \quad (5.86)$$

Once more, we remark that these estimates hold for all choice of g, χ fulfilling the conditions (h3). Let g_m and χ_m be positive defined, verifying the required hypothesis and such that: $g_m > |g|$ and $\chi_m > |\chi|$. For $j \neq j'$, the Cauchy-Schwarz inequality implies

$$\begin{aligned} \left| \int \frac{dk}{2\pi h} g(k) \langle \chi \psi_j(k, \cdot, t), \psi_{j'}(k, \cdot, t) \rangle_{L^2(\mathbb{R})} \right| &\leq \int \int \frac{dk dx}{2\pi h} g_m(k) \chi_m(x) |\psi_j(k, x, t) \psi_{j'}(k, x, t)| \\ &\leq \left(\int \frac{dk}{2\pi h} g_m(k) \langle \chi_m \psi_j(k, \cdot, t), \psi_j(k, \cdot, t) \rangle_{L^2(\mathbb{R})} \right)^{\frac{1}{2}} \left(\int \frac{dk}{2\pi h} g_m(k) \langle \chi_m \psi_{j'}(k, \cdot, t), \psi_{j'}(k, \cdot, t) \rangle_{L^2(\mathbb{R})} \right)^{\frac{1}{2}} \\ &\leq \tilde{\mathcal{O}}(\varepsilon^{\frac{1}{2}}). \end{aligned}$$

■

5.3 Remainder terms and proof of Theorem 2.1

We next consider the terms $\mathcal{J}_1(t)$ and $\mathcal{J}_2(t)$ in (5.53) in the limit $h \rightarrow 0$. To this aim, an asymptotic formula for the difference: $1 - \mu(t)$ is needed.

Lemma 5.4 *With the assumptions (h1)-(h4), the function $\mu(t)$, defined in (5.9), is such that*

$$\mu(t) = \left| \frac{\alpha_t}{\alpha_0} \right|^{\frac{3}{2}} \left(1 + \tilde{\mathcal{O}}(\varepsilon) \right) = 1 + \mathcal{O}(h). \quad (5.87)$$

Proof. From (5.9) and (5.65), our function writes as

$$\mu(t) = \frac{\|G(0)\|_{L^2(\mathbb{R})}}{\|G(t)\|_{L^2(\mathbb{R})}} e^{-i \int_0^t \frac{\text{Im} \langle G(s), \partial_s G(s) \rangle}{|G(s)|_2^2} ds} + \tilde{\mathcal{O}}(\varepsilon). \quad (5.88)$$

As $h \rightarrow 0$, an approximation of $\text{Im} \langle G(s), \partial_s G(s) \rangle$ is computable starting from the relation (4.12) taken with: $E = E_{res}^h = E(s)$ and $\alpha = \alpha_s$; this gives

$$\text{Im} \langle G(s), \partial_s G(s) \rangle = -\frac{1}{h\alpha_s} \text{Im} \frac{\dot{E}(s) \partial_2 M(E(s), E(s))}{M^2(E(s), E(s))} + \tilde{\mathcal{O}}(\varepsilon),$$

where ∂_2 denotes the derivative w.r.t the second variable. A relation for $\dot{E}(t)$ follows by taking the time derivative of (4.4),

$$\dot{E}(t) = \frac{\dot{\alpha}_t}{\alpha_t} \frac{G^{E(t)}(c, c)}{\partial_E G^E(c, c)|_{E(t)}}. \quad (5.89)$$

The r.h.s. of (5.89) is further developed by using (4.9); this leads to: $\dot{E}(t) = \frac{\dot{\alpha}_t |\alpha_t|}{2} + \mathcal{O}\left(e^{-\frac{|\alpha_t|}{h} d(c, \{a, b\})}\right)$.

Thus, $\dot{E}(t)$ is real, out of exponentially small terms, and the size of $\text{Im} \langle G(s), \partial_s G(s) \rangle$ is determined by the imaginary part of $M^{-2}(E(s), E(s)) \partial_2 M(E(s), E(s))$. According to (4.10), this quantity expresses as

$$\frac{\partial_2 M(E(s), E(s))}{M^2(E(s), E(s))} = \frac{\frac{1}{2} + \tilde{\mathcal{O}}\left(e^{-\frac{|\alpha_s|}{h} d(c, \{a, b\})}\right)}{\left| -\frac{\alpha_s^2}{2} + \tilde{\mathcal{O}}\left(e^{-\frac{|\alpha_s|}{h} d(c, \{a, b\})}\right) \right|^2} = -\frac{2}{\alpha_s^4} + \tilde{\mathcal{O}}(\varepsilon).$$

We finally get: $\text{Im} \langle G(s), \partial_s G(s) \rangle = \tilde{\mathcal{O}}\left(e^{-\frac{|\alpha_s|}{h} d(c, \{a, b\})}\right)$. It follows that

$$\mu(t) = \frac{\|G(0)\|_{L^2(\mathbb{R})}}{\|G(t)\|_{L^2(\mathbb{R})}} \left(1 + \tilde{\mathcal{O}}(\varepsilon) \right). \quad (5.90)$$

Since the Green's functions norms can be expressed in terms of $(h\alpha_t M(E(t), E(t)))^{-1}$ (we refer to (4.11)), the above ratio further expands as

$$\frac{\|G(0)\|_{L^2(\mathbb{R})}}{\|G(t)\|_{L^2(\mathbb{R})}} = \left| \frac{\alpha_t}{\alpha_0} \right|^{\frac{3}{2}} + \tilde{\mathcal{O}}(\varepsilon). \quad (5.91)$$

This result, together with the assumption (2.15), leads to (5.87). ■

The integral $\mathcal{J}_1(t)$ has the form

$$\mathcal{J}_1(t) = |1 - \mu(t)|^2 \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} \int \frac{dk}{2\pi h} g(k) |C(k, t)|^2. \quad (5.92)$$

According to (5.34)-(5.35), (4.11), (5.90), and using the exponential estimates for $G(t)$ outside $\text{supp } \chi$, this can be rephrased as

$$\mathcal{J}_1(t) = \left| 1 - \left| \frac{\alpha_t}{\alpha_0} \right|^{\frac{3}{2}} \right|^2 g\left(\lambda_t^{\frac{1}{2}}\right) (1 + \mathcal{O}(|\theta_0|)) + \tilde{\mathcal{O}}\left(e^{-\frac{\tau}{h}}\right), \quad (5.93)$$

for a suitable $\tau > 0$ and $d(c, \{a, b\}) = c - a$, otherwise we have: $\mathcal{J}_1 = \mathcal{O}\left(e^{-\frac{\beta}{h}}\right)$. The second remainder is a crossing term (see definition: 5.52); in Lemma 5.3 it has been shown that: $\mathcal{J}_2 = \mathcal{O}\left(\tilde{a}^{\frac{1}{2}} \tilde{\mathcal{J}}_1^{\frac{1}{2}}\right)$ where the variables \tilde{a} and $\tilde{\mathcal{J}}_1$ are the principal contribution and the first remainder associated with a suitable couple of auxiliary data $\tilde{g}, \tilde{\chi}$. If we assume $d(c, \{a, b\}) = b - c$, we have: $\mathcal{J}_2 \sim \tilde{a} \cdot \tilde{\mathcal{J}}_1 = \mathcal{O}\left(e^{-\frac{\beta}{h}}\right)$. When $d(c, \{a, b\}) = c - a$, this term is explicitly given by

$$\mathcal{J}_2(t) = 2 \operatorname{Re} \mu(t) (1 - \mu^*(t)) \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} \int \frac{dk}{2\pi h} g(k) \beta(k, t) C^*(k, t). \quad (5.94)$$

After an integration by part, we get

$$\mathcal{J}_2(t) = 2 \operatorname{Re} \mu(t) (1 - \mu^*(t)) \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} [I + II], \quad (5.95)$$

$$I = \int \frac{dk}{2\pi h} g(k) C(k, 0) C^*(k, t) e^{-\frac{i}{\varepsilon} \int_0^t (E(\sigma) - k^2) d\sigma}, \quad (5.96)$$

$$II = \frac{i}{\varepsilon} \int \frac{dk}{2\pi h} g(k) \int_0^t C(k, s) C^*(k, t) (E(s) - k^2) e^{-\frac{i}{\varepsilon} \int_s^t (E(\sigma) - k^2) d\sigma} ds. \quad (5.97)$$

The small- h behaviour of I and II is investigated using a path-deformation argument and following the same line as in (5.64). As before, $\mathcal{C}_{\frac{h}{d_0}}^+(\lambda_0)$ denotes the semicircle of center λ_0 , radius $\frac{h}{d_0}$, but now we fix $\operatorname{Im} \mathcal{C}_{\frac{h}{d_0}}^+(\lambda_0) > 0$. Replacing $C(k, 0) C^*(k, t)$ with $C(k, 0) C^*(k^*, t)$, we define a meromorphic function in a neighbourhood of λ_0 with simple poles at $k^2 = E(0), E^*(t)$. Thus, the first integral is

$$I = - \int_{\mathcal{C}_{\frac{h}{d_0}}^+(\lambda_0)} \frac{dE}{4\pi h E^{\frac{1}{2}}} g\left(E^{\frac{1}{2}}\right) \mathcal{K}(E, 0, t) e^{-\frac{i}{\varepsilon} \int_0^t (E(\sigma) - E) d\sigma} dE + 2\pi i \operatorname{Res}_1(E^*(t)) + \tilde{\mathcal{O}}(\varepsilon), \quad (5.98)$$

where $\operatorname{Res}_1(E^*(t))$ is the residue at $E^*(t)$, while $\mathcal{K}(E, s, t)$ denotes

$$\mathcal{K}(E, s, t) = C\left(E^{\frac{1}{2}}, s\right) C^*\left((E^*)^{\frac{1}{2}}, t\right).$$

Since $\operatorname{Im}(E(\sigma) - k^2) < 0$ and $\mathcal{K}(E, s, t) = \tilde{\mathcal{O}}(\varepsilon)$ for $E \in \mathcal{C}_{\frac{h}{d_0}}^+(\lambda_0)$ (according to (5.26)), we have

$$I = 2\pi i \operatorname{Res}_1(E^*(t)) + \tilde{\mathcal{O}}(\varepsilon). \quad (5.99)$$

Computing the residue when $h \rightarrow 0$, $E(t)$ can be replaced with its limit value λ_t , excepting those parts of the function where the difference $E^*(t) - E(0)$ appears. In this case we use: $E(t) = \lambda_t - i\Gamma_t$. Out of exponentially small terms, the result is

$$\operatorname{Res}_1(E^*(t)) = \frac{h\alpha_0\alpha_t}{4\pi\lambda_t^{\frac{1}{2}}} g\left(\lambda_t^{\frac{1}{2}}\right) M(\lambda_t, \lambda_0) M^*(\lambda_t, \lambda_t) \frac{\left|\tilde{\psi}_-(\lambda_t^{\frac{1}{2}}, c)\right|^2}{E^*(t) - E(0)} e^{-\frac{i}{\varepsilon} \int_0^t (E(\sigma) - E^*(t)) d\sigma}. \quad (5.100)$$

Using (4.10) and (5.25), it follows

$$2\pi i \operatorname{Res}_1(E^*(t)) = i |\alpha_0| (\alpha_t^2 + \alpha_0\alpha_t) g\left(\lambda_t^{\frac{1}{2}}\right) \frac{e^{-\frac{i}{\varepsilon} \int_0^t (E(\sigma) - E^*(t)) d\sigma} \Gamma_t}{E^*(t) - E(0)} \frac{1}{2} (h + \mathcal{O}(|h\theta_0|)). \quad (5.101)$$

Adopting the same notation, the second contribution writes as

$$II = -\frac{i}{\varepsilon} \int_{\mathcal{C}_{\frac{h}{d_0}}^+(\lambda_0)} \frac{dE}{4\pi h E^{\frac{1}{2}}} g(E^{\frac{1}{2}}) \int_0^t \mathcal{K}(E, s, t) (E(s) - E) e^{-\frac{i}{\varepsilon} \int_s^t (E(\sigma) + E) d\sigma} ds + 2\pi i \operatorname{Res}_2(E^*(t)) + \tilde{\mathcal{O}}(\varepsilon) \quad (5.102)$$

Proceeding as the previous Section (see the estimate of $\mathcal{S}_2^h(t)$), the integral over $\mathcal{C}_{\frac{h}{d_0}}^+(\lambda_0)$ is bounded as $\mathcal{O}(\varepsilon^{\frac{1}{2}})$, while the residue in $E^*(t)$ is given, out of exponentially small terms, by

$$Res_2(E^*(t)) = -i \frac{h}{4\pi} g(\lambda_t^{\frac{1}{2}}) \frac{\Gamma_t}{\varepsilon} (1 + \mathcal{O}(|\theta_0|)) \int_0^t f(s, t) e^{-\frac{i}{\varepsilon} \varphi(s, t)} ds,$$

$$f(s, t) = (\alpha_s \alpha_t^2 + \alpha_s^2 \alpha_t) e^{-\frac{1}{\varepsilon} \int_s^t (\Gamma_\sigma + \Gamma_t) d\sigma}; \quad \varphi(s, t) = \int_s^t (\lambda_\sigma - \lambda_t) d\sigma.$$

If $d(c, \{a, b\}) = c - a$, the factor $\frac{\Gamma_t}{\varepsilon}$ is $\mathcal{O}(1)$, and the size of II is determined by the oscillatory integral. To this concern, we notice that: $\partial_s \varphi(s, t) = \lambda_t - \lambda_s$; according to the definition of λ_t , the stationary points of $\varphi(s, t)$ are defined by the equation

$$\alpha_s - \alpha_t = 0. \quad (5.103)$$

It follows from (h2) that the set of the 's' fulfilling the condition (5.103) does not have accumulation points in $[0, t]$, forming a subset of finite cardinality. It means that $s \rightarrow \varphi(s, t)$ have a finite number of stationary points $\{s_j(t)\}_{j=1}^{N(t)} \subset [0, t]$, depending on t . Since $s \rightarrow f(s, t)$ is a regular function (with $\alpha_s, \Gamma_s \in C^\infty$) the stationary phase method applies with: $|\partial_s^{j+1} \varphi(s, t)| = |\partial_s^j E_R(s)| \gtrsim |\partial_t^j \alpha(t)| > 0$ for some $j \in \{1, \dots, k\}$. This yields: $\int_0^t f(s, t) e^{-\frac{i}{\varepsilon} \varphi(s, t)} ds = \mathcal{O}\left(\varepsilon^{\frac{1}{j+1}}\right)$ and

$$Res_2(E^*(t)) = \mathcal{O}\left(\varepsilon^{\frac{1}{j+1}}\right), \quad (5.104)$$

and uniformly w.r.t. the time. According to the definition (5.95) and the expansions (5.99), (5.101), (5.104), we get

$$\begin{aligned} \mathcal{J}_2(t) = 2 \operatorname{Re} \left[i \mu(t) (1 - \mu^*(t)) \langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})} g\left(\lambda_t^{\frac{1}{2}}\right) \frac{\Gamma_t}{2} \times \right. \\ \left. \times |\alpha_0| (\alpha_t^2 + \alpha_0 \alpha_t) \frac{e^{-\frac{i}{\varepsilon} \int_0^t (E(\sigma) - E^*(t)) d\sigma}}{E^*(t) - E(0)} (h + \mathcal{O}(|h \theta_0|)) \right] \end{aligned}$$

Expanding $\langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})}$ and $\mu(t)$ with (4.11), (5.87) and using $E(t) = \lambda_t - i\Gamma_t + \tilde{\mathcal{O}}(\varepsilon)$, leads to

$$\mathcal{J}_2(t) = \operatorname{Re} 2i \left(1 - \left| \frac{\alpha_t}{\alpha_0} \right|^{\frac{3}{2}} \right) \frac{\Gamma_t}{\varepsilon} g\left(\lambda_t^{\frac{1}{2}}\right) \frac{\mathcal{T}(t)}{\frac{\lambda_t - \lambda_0}{\varepsilon} - i \frac{(\Gamma_t + \Gamma_0)}{\varepsilon}}, \quad (5.105)$$

$$\mathcal{T}(t) = \frac{|\alpha_0| \alpha_t^2 + \alpha_0^2 |\alpha_t|}{(\alpha_0 \alpha_t)^{\frac{3}{2}}} e^{-\frac{1}{\varepsilon} \int_0^t (\Gamma_\sigma + \Gamma_t) d\sigma} e^{-\frac{i}{\varepsilon} \int_0^t (\lambda_\sigma - \lambda_t) d\sigma}. \quad (5.106)$$

When $d(c, \{a, b\}) = c - a$, Γ_t is $\mathcal{O}(\varepsilon)$ and the small- h behaviour of this quantity is determined by the ratio: $\frac{\lambda_t - \lambda_0}{\varepsilon}$. In particular, for $\lambda_t \neq \lambda_0$, one has: $Res_1(E^*(t)) = \mathcal{O}(\varepsilon)$. However, if: $E_R(0) - E_R(t) \sim \mathcal{O}(\varepsilon)$, a boundary layer contribution is expected.

Proof of Theorem 2.1. *i)* This first point is a rewriting of the result of Proposition 4.1.
ii) The second point comes from the decomposition (5.23) and the results of Lemmas 5.2 and 5.3. The reduced equation for the main contribution $a(t)$ is obtained in (5.79) for $d(c, \{a, b\}) = c - a$, while this variable is exponentially small, according to the estimate (5.82), when $d(c, \{a, b\}) = b - c$.
iii) Once the small- h behaviour of the factors $\mu(t)$, $(1 - \mu^*(t))$ and $\langle \chi G(t), G(t) \rangle_{L^2(\mathbb{R})}$ is taken into account, the last point is a consequence of (5.93), (5.95), (5.99) and (5.105)-(5.106). ■

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